Unit I

Matrices :-

Formulas 1) Characteristic equation of a 2 x 2 matrix

$\lambda^2 - S_1\lambda + S_2 = 0$

Where S_1 = Sum of diagonal elts

$S_2 = |A|$.

2) Characteristic equation of a 3 x 3 matrix

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where S_1 = sum of diagonal elts
S_2 = sum of minors
$S_3 = |A|$

3) In general $|A - \lambda I| = 0$ is called the characteristic eqn of a matrix.

Problems

① Find the characteristic eqn of a matrix

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}.$$

Soln

Method (1)

Formula : Char. eqn is $\lambda^2 - S_1\lambda + S_2 = 0$.

$S_1 = 1 + 4 = 5$ (sum of diagonal elts.)

$$S_2 = \begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix}$$

$= 4 + 6 = 10$.

$\therefore$ char eqn is $\lambda^2 - 5\lambda + 10 = 0$.

Method(2) $|A - \lambda I| = 0$ is called characteristic equation.

$$\left| \left(\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)(4-\lambda) + 6 = 0.$$

$$\lambda^2 - \lambda - 4\lambda + 4 + 6 = 0.$$

$\therefore \lambda^2 - 5\lambda + 10 = 0$. is required char. eqn.

Problem(2) Find the characteristic equation of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Soln:-

Method(1) characteristic eqn is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0.$$

Eigen Values and Eigen Vectors

Let A be a given matrix and X be any non zero vector satisfies the eqn $A\lambda = X\lambda$ for any scalar λ. Then λ is called eigen value of a matrix A and X is called the eigen vectr of a matrix A.

Formula : $\boxed{(A - \lambda I)X = 0}$

- Eigen values :- Roots of the characteristic eqn is called the eigen values.

Problem ① Find the eigen values of $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Soln :-

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}.$$

Char. eqn : $\lambda^2 - S_1\lambda + S_2 = 0.$

$$S_1 = 4 + 2 = 6.$$

$$S_2 = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 8 - 3$$

$$= 5.$$

$S_1 = 3+3+3 = 9.$

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= (9-1) + (9-1) + (9-1)$$

$$= 8 + 8 + 8$$

$S_2 = 24.$

$S_3 = |A|$

$$= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3\left[9-1\right] - 1\left[3+1\right] + 1\left[-1+3\right]$$

$$= 3(8) - 1(4) + 1(2)$$

$$= 24 - 4 + 2$$

$S_3 = 22.$

$\therefore$ Char. eqn is $\lambda^3 - 9\lambda^2 + 24\lambda - 22 = 0.$

② Find the characteristic values for $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Soln

Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Characteristic equation: $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$

S_1 = Sum of diagonal elements

$= 1+2+3$

$= 6$

S_2 = Sum of minors

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$= (6-2) + (3+2) + (2-0)$

$= 4+5+2$

$= 11$

$S_3 = |A|$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$$

$= 1[6-2] - 0 - 1[2-4]$

$= 1(4) - 1(-2)$

$= 4+2$

$\boxed{S_3 = 6}$

$S_1 = 6$
$S_2 = 11$
$S_3 = 6$

$\therefore$ char. eqn : $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

To find characteristic Values:

Solve $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Put $\lambda = 1$, $1 - 6 + 11 - 6 = 0$

$\therefore \lambda = 1$ is a root

By synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\therefore 1\lambda^2 - 5\lambda + 6 = 0.$

$\lambda^2 - 5\lambda + 6 = 0.$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2(a)}$$

$a = 1, b = -5, c = 6.$

$$\lambda = \frac{5 \pm \sqrt{25 - 4(6)(1)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$\lambda = \frac{5 \pm \sqrt{1}}{2}$$

$\therefore \lambda = \frac{5+1}{2}$, $\lambda = \frac{5-1}{2}$

$\lambda = 3$, $\lambda = 2$

$\therefore$ Characteristic values are 1, 2, 3.

$S_1 = -2 + 1 + 0 = -1.$

$S_2 =$ Determinant of minors

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$= -12 + (-3) + (-2 \mp 4)$

$= -12 - 3$ [illegible] $2 - 6$

$= -21$

$$S_3 = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$= -2\left[0 - 12\right] - 2\left[0 - 6\right] - 3\left[-4 + 1\right]$

$= 24 + 12 + 9$

$= 45.$

$\therefore$ Char. eqn is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

③ Find the characteristic eqn of $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Soln :-

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

S_1 = Sum of diagonal elts

$= 1+1$

$= 2.$

$S_2 = |A|$

$$= \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$= 1+0$

$= 1.$

$\therefore$ Char. eqn is $\lambda^2 - S_1\lambda + S_2 = 0.$

$\lambda^2 - 2\lambda + 1 = 0.$

④ Find the characteristic eqn of $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$.

Soln

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Char. eqn $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$

① Reduce the quadratic form $6x^2+3y^2+3z^2-4xy-2yz+4xz$ into canonical form by an orthogonal transformation.

Soln:-

$$A = \begin{array}{c|ccc} & x & y & z \\ \hline x & \text{coeff } x^2 & \frac{1}{2}\text{coeff } xy & \frac{1}{2}\text{coeff } xz \\ y & \frac{1}{2}\text{coeff } yx & \text{coeff } y^2 & \frac{1}{2}\text{coeff } yz \\ z & \frac{1}{2}\text{coeff } zx & \frac{1}{2}\text{coeff } zy & \text{coeff } z^2 \end{array}$$

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

To find the char. eqn

Formula: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = sum of diagonal elements

$= 6+3+3$

$S_1 = 12$

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$= (9-1) + (18-4) + (18-4)$

$= 8 + 14 + 14$

$= 36.$

$$S_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 6(9-1) - (-2)[-6+2] + 2[2-6]$$

$= 6(8) + 2(-4) + 2(-4)$

$= 48 - 16$

$= 32.$

Char. eqn is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

Solving we get $\lambda_1 = 8$

$\lambda_2 = 2$ are eigen values.

$\lambda_3 = 2$.

To find Eigen Vector

Formula $(A - \lambda I)X = 0$

$$\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow ①$$

case (i) Suppose $\lambda = 8$.

From ①
$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 - 5x_2 - x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0 \end{aligned}\right\} \text{Different eqns.}$$

consider
$$-2x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 - 5x_2 - x_3 = 0$$

$$\frac{x_1}{2+10} = \frac{-x_2}{2+4} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}.$$

$$\boxed{\therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ is corresponding eigen vector.}}$$

Case (ii) Suppose $\lambda = 2$

From ① $\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 4x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + x_2 - x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}\right\} \text{Same eqns.}$$

consider $-2x_1 + x_2 - x_3 = 0$

Put $x_1 = 0$, $x_2 = 1$

$1 - x_3 = 0 \Rightarrow \boxed{x_3 = 1}$

$\therefore X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is second eigen vector.

Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be the third eigen vector.

$X_1^T X_3 = 0 \Rightarrow (2 \ -1 \ 1) \begin{pmatrix} l \\ m \\ n \end{pmatrix} = 0 \Rightarrow 2l - m + n = 0$

$X_2^T X_3 = 0 \Rightarrow (0 \ 1 \ 1) \begin{pmatrix} l \\ m \\ n \end{pmatrix} = 0 \Rightarrow 0l + m + n = 0$

$$\frac{l}{-1-1} = \frac{-m}{2-0} = \frac{n}{2-0}$$

$$\frac{l}{-2} = \frac{-m}{2} = \frac{n}{2}$$

$$\frac{l}{-1} = \frac{m}{-1} = \frac{n}{1}$$

$X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is third eigen vector

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

<u>checking</u>:

$$x_1^T x_2 = (2\ -1\ 1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0-1+1 = 0 \checkmark$$

$$x_2^T x_3 = (0\ 1\ 1)\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 0-1+1 = 0 \checkmark$$

$$x_3^T x_1 = (-1\ -1\ 1)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = -2+1+1 = 0 \checkmark$$

$$N = \begin{bmatrix} \frac{2}{\sqrt6} & 0 & -\frac{1}{\sqrt3} \\ \frac{-1}{\sqrt6} & \frac{1}{\sqrt2} & -\frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt2} & \frac{1}{\sqrt3} \end{bmatrix}$$

$\sqrt{2^2+(-1)^2+1^2} = \sqrt6$

$\sqrt{0^2+1^2+1^2} = \sqrt2$

$\sqrt{1^2+1^2+1^2} = \sqrt3$.

$$N^T = \begin{bmatrix} 2/\sqrt6 & -1/\sqrt6 & 1/\sqrt6 \\ 0 & 1/\sqrt2 & 1/\sqrt2 \\ -1/\sqrt3 & -1/\sqrt3 & 1/\sqrt3 \end{bmatrix}$$

$$D = N^TAN = \begin{pmatrix} 2/\sqrt6 & -1/\sqrt6 & 1/\sqrt6 \\ 0 & 1/\sqrt2 & 1/\sqrt2 \\ -1/\sqrt3 & -1/\sqrt3 & 1/\sqrt3 \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2/\sqrt6 & 0 & -1/\sqrt3 \\ -1/\sqrt6 & 1/\sqrt2 & -1/\sqrt3 \\ 1/\sqrt6 & 1/\sqrt2 & 1/\sqrt3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Canonical Form $= 8y_1^2 + 2y_2^2 + 2y_3^2$.

② Reduce the Quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonical Form.

Soln

$$A = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{ccc} x & y & z \\ \text{coeff } x^2 & \frac{1}{2}\text{coeff } xy & \frac{1}{2}\text{coeff } xz \\ \frac{1}{2}\text{coeff } yx & \text{coeff } y^2 & \frac{1}{2}\text{coeff } yz \\ \frac{1}{2}\text{coeff } zx & \frac{1}{2}\text{coeff } zy & \text{coeff } z^2 \end{array}$$

$$A = \begin{pmatrix} 3 & -2/2 & 2/2 \\ -2/2 & 5 & -2/2 \\ 2/2 & -2/2 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

To find characteristic eqn

Char.eqn : $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = sum of diagonal elements $= 3+5+3 = 11$

$$S_2 = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$= (15-1) + (9-1) + (15-1)$

$= 14 + 8 + 14$

$= 36$

$$S_3 = |A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 3\left[15-1\right] - (-1)\left[-3+1\right] + 1\left[1-5\right]$$

$= 3(14) + 1(-2) + 1(-4)$

$= 42 - 2 - 4$

$= 36.$

∴ char.eqn is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

Solving we get $\lambda_1 = 2$

$\lambda_2 = 3$

$\lambda_3 = 6$.

To find Eigen vectors

Formula: $(A - \lambda I)(X) = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow ①$$

Case (i) Suppose $\lambda = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + 3x_2 - x_3 &= 0 \end{aligned}\right\} \text{Different eqns.}$$

$$x_1 - x_2 + x_3 = 0$$

consider $x_1 - x_2 + x_3 = 0$

$-x_1 + 3x_2 - x_3 = 0$

$$\frac{x_1}{1-3} = \frac{-x_2}{-1+1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}.$$

$\therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is first eigen vector.

case(ii) Suppose $\lambda = 3$

From ① $$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 0.x_1 - x_2 + x_3 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + 0.x_3 &= 0. \end{aligned}\right\} \text{Different eqns.}$$

Consider $$\begin{aligned} 0.x_1 - x_2 + x_3 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0. \end{aligned}$$

$$\frac{x_1}{1-2} = \frac{-x_2}{0+1} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is 2nd eigen vector.

case(iii) Suppose $\lambda = 6$.

From ① $$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -3x_1 - x_2 + x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \\ x_1 - x_2 - 3x_3 &= 0 \end{aligned}\right\} \text{Different eqns.}$$

Consider $$\begin{aligned} -3x_1 - x_2 + x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \end{aligned}$$

$$\frac{x_1}{1+1} = \frac{-x_2}{3+1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$\therefore X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is 3rd eigen vector

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

checking

$X_1^T X_2 = (-1\ 0\ 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -1+0+1 = 0$ ✓

$X_2^T X_3 = (1\ 1\ 1)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 1-2+1 = 0$ ✓

$X_3^T X_1 = (+1\ -2\ 1)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1-0+1 = 0$ ✓

$$N = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$\sqrt{(-1)^2+0^2+1^2} = \sqrt{2}$

$\sqrt{1^2+1^2+1^2} = \sqrt{3}$

$\sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$.

$$N^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$D = N^T A N$

$$= \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Canonical Form $= 2y_1^2 + 3y_2^2 + 6y_3^2$

Properties of Eigen Values & Eigen Vectors.

1) Sum of eigenvalues = sum of diagonal elts.

2) Product of eigenvalues = $|A|$.

3) A and A^T have same eigen values.

4) If λ is an eigen value for A then $\frac{1}{\lambda}$ is an eigen value for A^{-1}.

Problems

① Find the sum and product of the eigen values of the matrix $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Soln

Sum of eigen values = Sum of diagonal elts

$= -1-1-1$

$= -3.$

Product of eigen values = $|A|$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= -1(1-1) - 1(-1-1) + 1(1+1)$

$= 2 + 2$

$= 4.$

② The product of two eigen values of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ ~~are~~ ~~2 and 8.~~ is 16. Find the third eigenvalue.

Soln Let $\lambda_1, \lambda_2, \lambda_3$ be eigen values.

$\lambda_1 \lambda_2 = 16.$

$\lambda_3 = ?$

$\lambda_1 + \lambda_2 + \lambda_3 = 6 + 3 + 3$

$$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 6\left[9-1\right] + 2\left[-6+2\right] + 2\left[2-6\right]$

$= 6(8) + 2(-4) + 2(-4)$

$= 48 - 8 - 8$

$= 32.$

$\therefore \quad \lambda_3 = \frac{32}{16} = 2.$

Third eigen value $= 2.$

② Two of the eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigen values of A^{-1}.

Soln

$\lambda_1 = 3, \quad \lambda_2 = 6, \quad \lambda_3 = ?$

W.K.T $\lambda_1 + \lambda_2 + \lambda_3 = 3+5+3$

$= 11.$

$\therefore \lambda_3 = 11 - (\lambda_1 + \lambda_2)$

$= 11 - (3+6)$

$= 11 - 9$

$\boxed{\lambda_3 = 2.}$

$\therefore$ 3, 6, 2 are eigen values of A.

$\Rightarrow \frac{1}{3}, \frac{1}{6}, \frac{1}{2}$ are eigen values of A^{-1}.

Eigen Vectors

Formula $(A-\lambda I).X = 0$

Problem ① Find the eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$$

Soln :-

The char. eqn is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$$S_1 = 2+1-3 = 0.$$

$$S_2 = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -7 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (-3-2) + (-6+0) + (2-4)$$

$$= -5 + (-6) + (-2)$$

$$= -13.$$

$$S_3 = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{vmatrix}$$

$$= 2\left[-3-2\right] - 2\left[-6+7\right]$$

$$= -10 - 2$$

$$= -12.$$

Char.eqn is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$$\lambda^3 - 0.\lambda^2 - 13\lambda + 12 = 0.$$

$$\lambda^3 - 13\lambda + 12 = 0. \longrightarrow ①$$

Clearly $\lambda = 1$ is a root of ①

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & +0 & -13 & +12 \\ & 0 & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & \boxed{0} \end{array}$$

$$\lambda^2 + \lambda - 12 = 0.$$

$$\begin{array}{c|c} 4 & -3 \\ \hline \lambda & \lambda \end{array}$$

$$(\lambda + 4)(\lambda - 3) = 0.$$

$\lambda = -4, 3.$

∴ Eigen values are $1, 3, -4$.

Eigen vector Formula $[A - \lambda I]X = 0$

$$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \text{②}$$

Case (i) $\lambda = 1.$

$$\begin{bmatrix} \cancel{2}1 & 2 & 0 \\ 2 & 0 & 1 \\ -7 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 0.x_3 = 0. \longrightarrow \text{①}$$

$$2x_1 + 0.x_2 + 1.x_3 = 0 \longrightarrow \text{②}$$

$$-7x_1 + 2x_2 - 4x_3 = 0. \longrightarrow \text{③}$$

Solving ① & ②

$$\frac{x_1}{2-0} = \frac{-x_2}{1-0} = \frac{x_3}{0-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-4}.$$

$$\therefore X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$ is corresponding eigen vector.

Case (ii) $\lambda = 3.$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -7 & 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$-x_1 + 2x_2 + 0.x_3 = 0 \longrightarrow ④$$

$$2x_1 - 2x_2 + x_3 = 0. \longrightarrow ⑤$$

$$-7x_1 + 2x_2 - 6x_3 = 0. \longrightarrow ⑥.$$

$$\frac{x_1}{2} = \frac{-x_2}{-7} = \frac{x_3}{2-4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}.$$

$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is corresponding eigen vector

Case (iii) $\lambda = -4.$

$$\begin{bmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ -7 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$6x_1 + 2x_2 + 0.x_3 = 0 \longrightarrow ⑦$$

$$2x_1 + 5x_2 + 1.x_3 = 0. \longrightarrow ⑧$$

$$-7x_1 + 2x_2 + 1.x_3 = 0. \longrightarrow ⑨$$

solving ⑦ & ⑧.

$$\frac{x_1}{2} = \frac{-x_2}{6} = \frac{x_3}{30-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-6} = \frac{x_3}{26}.$$

$$\frac{x_1}{1} = \frac{x_2}{-3} = \frac{x_3}{13}.$$

Case (iii) When $\lambda = 3$.

$$-x_1 + x_2 + x_3 = 0.$$

$$x_1 - x_2 + x_3 = 0.$$

$$0.x_1 + 0x_2 - 2x_3 = 0.$$

$$\frac{x_1}{1+1} = \frac{-x_2}{-1-1} = \frac{x_3}{1-1}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0}.$$

$\therefore$ $X_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ is corresponding eigen vector.

Cayley Hamilton Theorem

Statement :- Every square matrix satisfies its own characteristic equation.

Problems ① Verify cayley hamilton thm for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Soln :-

char. eqn $\lambda^2 - S_1\lambda + S_2 = 0$.

$$S_1 = 1 - 1 = 0.$$

$$S_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5.$$

$\therefore$ Char. eqn is $\lambda^2 \overline{\cancel{+}} 5 = 0$.

Verification :-

To prove $A^{2} \overset{-}{\cancel{+}} 5I = 0$

L.H.S $A^{2}+5I$.

$$A^{2} = A \cdot A$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

$$5I = 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\therefore A^{2} \overset{-}{\cancel{+}} 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0$$

L.H.S. = R.H.S

Problem ② Verify cayley hamilton thm for the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence evaluate the matrix eqn $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 + 2A - I$.

Soln

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

$S_1 = 2 + 1 + 2$

$= 5$

$S_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= (2-0) + (4-1) + (2-0)$

$= 2 + 3 + 2$

$S_2 = 7.$

$S_3 = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$

$= 2(2-0) - 1(0-0) + 1(0-1)$

$= 4 - 1$

$= 3.$

$\therefore$ char. eqn is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$.

Verifycarion of C-H thm:-

To prove. $A^3 - 5A^2 + 7A - 3I = 0$.

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} : \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}.$$

$$-5A^2 = \begin{bmatrix} -25 & -20 & -20 \\ 0 & -5 & 0 \\ -20 & -20 & 25 \end{bmatrix}$$

$$+7A = \begin{bmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{bmatrix}.$$

$$-3I = -3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Adding above all

$$A^3 - 5A^2 + 7A - 3I = O.$$

Let $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 + 2A - I$.

$$\therefore f(\lambda) = \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 + 2\lambda - 1.$$

$$\begin{array}{r|l} & \lambda^5 + \lambda \\ \hline \lambda^3 - 5\lambda^2 + 7\lambda - 3 & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 + 2\lambda - 1. \\ & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 \\ \cline{2-2} & (-) \qquad \lambda^4 - 5\lambda^3 + 8\lambda^2 + 2\lambda - 1 \\ & \qquad \lambda^4 - 5\lambda^3 + 7\lambda^2 - 3\lambda \\ \cline{2-2} & \qquad \lambda^2 + 5\lambda - 1 \\ \cline{2-2} \end{array}$$

$$\therefore f(A) = O + (A^2 + 5A - I)$$

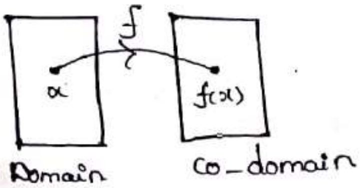
$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 5 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Unit - II

Differential Calculus

Defn:- (Function)

A function is a rule which assigns a unique value 'f(x)' in the co-domain for each value 'x' in a domain.



Defn:- (Real Valued Function)

A function whose co-domain is a real line is called real valued function.

Range :- Range of $f = \{ f(x) / x \in \text{Domain} \}$

Graph :- Graph of f is a subset in the $X \times Y$ plane such that, Graph of $f = \{ (x, f(x)) / x \in \text{Domain} \}$

One-to-One Function :-

If distinct elements have distinct images then f is 1-1 function.

Onto Function :-

A function is onto if Range f = codomain.

Four ways to represent Functions :-

1) Verbally

2) Visually

3) Graphically

4) Numerically.

Limit of a Function :-

$\lim_{x \to a} f(x) = l$ means that if we approach x is near to a, then $f(x)$ will be mach closer ~~to~~ to 'l'.

For example,

1) $\lim_{x \to 2} 3x = 6.$

2) $\lim_{x \to \frac{\pi}{2}} \sin x = 1.$

Problems

① Find $\lim\limits_{x \to 1} \frac{x^2-1}{x-1}$

Soln:-

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)}$$

$$= (x+1).$$

$$\therefore \lim_{x \to 1} \left(\frac{x^2-1}{x-1}\right) = \lim_{x \to 1} (x+1)$$

$$= 2.$$

② Find $\lim\limits_{t \to 0} \left(\frac{\sqrt{t^2+9} - 3}{t^2}\right)$

Soln:-

$$\lim_{t \to 0} \left[\frac{\sqrt{t^2+9} - 3}{t^2}\right] = \lim_{t \to 0} \left(\frac{\left(\sqrt{t^2+9} - 3\right) \cdot \left(\sqrt{t^2+9} + 3\right)}{t^2\left(\sqrt{t^2+9} + 3\right)}\right)$$

$$= \lim_{t \to 0} \left(\frac{\left(\sqrt{t^2+9}\right)^2 - (3)^2}{t^2\left(\sqrt{t^2+9} + 3\right)}\right)$$

$$= \lim_{t \to 0} \left[\frac{t^2+9-9}{t^2\left[\sqrt{t^2+9} + 3\right]}\right]$$

$$= \lim_{t \to 0} \left[\frac{1}{\sqrt{t^2+9} + 3}\right]$$

$$= \frac{1}{\sqrt{9} + 3}$$

$$= \frac{1}{3+3}.$$

$$= 1/6.$$

(iii) $f(x) = x\cos x$.

$f(-x) = -x \cdot \cos(-x)$

$= -x \cdot \cos x$ $\quad (\because \cos(-\theta) = \cos\theta)$

$= -f(x)$.

$\therefore f(x)$ is odd.

(iv) $f(x) = 1 - \sin x$.

$f(-x) = 1 - \sin(-x)$

$= 1 + \sin x$ $\quad (\because \sin(-\theta) = -\sin\theta)$

$f(-x) \neq -f(x)$

&

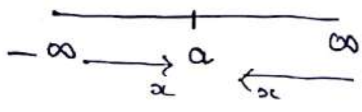
$f(-x) \neq f(x)$.

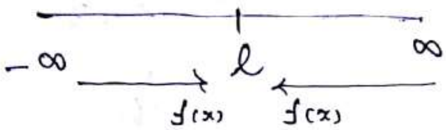
$\therefore f(x)$ is neither even nor odd.

Defn (Limit of a Function):-

Let f be a function of a real variable 'x' let a, l be two numbers. If f(x) approaches the value 'l' as x approaches 'c' we say 'l' is the limit of a function f(x) as x tends to a.

We write $\lim_{x\to a} f(x) = l$.





Squeece Theorem (or) Pinching Theorem (or)
Sandwich Theorem :-

If $f(x) \leq g(x) \leq h(x)$ when x is near 'a'

and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ then

$\lim_{x \to a} g(x) = L$.

Problems :-

① Find $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

Soln :-

By ordinary method

$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x \to 0} x^2 . \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

~~does not exist~~ since $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

$\therefore$ We cannot find limit using ordinary method.

We know that

$$-1 \leq \sin\frac{1}{x} \leq 1.$$

$$\therefore \quad -x^2 \leq x^2 \sin\frac{1}{x} \leq x^2;$$

$\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2$

$\therefore \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$. (By Sandwich thm)

Some special limits :-

(i) $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$

(iii) $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

Problems

① Find $\lim\limits_{x \to a} \left(\frac{x^{5/8} - a^{5/8}}{x^{1/3} - a^{1/3}}\right)$

Soln:-

$$\lim_{x \to a} \left(\frac{x^{5/8} - a^{5/8}}{x^{1/3} - a^{1/3}}\right) = \lim_{x \to a} \left[\left(\frac{x^{5/8} - a^{5/8}}{x - a}\right) \times \left(\frac{x - a}{x^{1/3} - a^{1/3}}\right)\right]$$

$$= \frac{\lim\limits_{x \to a} \left(\frac{x^{5/8} - a^{5/8}}{x - a}\right)}{\lim\limits_{x \to a} \left(\frac{x^{1/3} - a^{1/3}}{x - a}\right)}$$

$$= \frac{5/8 \; a^{5/8 - 1}}{1/3 \; a^{1/3 - 1}}$$

$$= \frac{15}{8} a^{\frac{5}{8} - 1 - 1/3 + 1}$$

$$= \frac{15}{8} a^{5/8 - 1/3}$$

$$= \frac{15}{8} a^{\frac{7}{24}}$$

② Find $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta}$

Soln:-

$$\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \left[\frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta}\right]$$

$$= \frac{\lim\limits_{\theta \to 0} \left(\dfrac{\sin\theta}{\theta}\right)}{\lim\limits_{\theta \to 0} \cos\theta}$$

$$= \frac{1}{1}$$

$$= 1.$$

③ Find $\lim\limits_{x \to \frac{\pi}{6}} \left(\dfrac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}\right)$

$$= \lim_{x \to \frac{\pi}{6}} \left[\frac{2\sin^2 x + 2\sin x - \sin x - 1}{2\sin^2 x - 2\sin x - \sin x + 1}\right]$$

$$= \lim_{x \to \frac{\pi}{6}} \left[\frac{2\sin x(\sin x + 1) - (\sin x + 1)}{2\sin x(\sin x - 1) - (\sin x - 1)}\right]$$

$$= \lim_{x \to \frac{\pi}{6}} \left[\frac{(\sin x + 1)(2\sin x - 1)}{(\sin x - 1)(2\sin x - 1)}\right]$$

$$= \lim_{x \to \frac{\pi}{6}} \left(\frac{\sin x + 1}{\sin x - 1}\right)$$

$$= \frac{\sin \pi/6 + 1}{\sin \pi/6 - 1}$$

$$= \frac{1/2 + 1}{1/2 - 1}$$

$$= \frac{3/2}{-1/2}$$

$$= 3/2 \times 2/-1$$

$$= -3.$$

④ Find $\lim_{x \to 0} (1+x)^{1/x}$

Soln

Formula $\lim_{n \to \infty} (1 + 1/n)^n = e.$

Put $x = 1/n.$

As $n \to \infty$, $x \to 0$.

$\therefore \lim_{x \to 0} (1+x)^{1/x} = \lim_{n \to \infty} (1 + 1/n)^n$

$= e.$

⑤ Find the least positive integer 'n' such that

$$\lim_{x\to 2} \frac{x^n - 2^n}{x-2} = 32.$$

Soln:-

We know that

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}.$$

Put $a=2$

$$\lim_{x\to 2} \frac{x^n - 2^n}{x-2} = n2^{n-1} = 32.$$

(i) $n2^{n-1} = 32$

$= 4 \times 8$

$= 4 \times 2^3$

$= 4 \times 2^{4-1}.$

$\therefore \boxed{n = 4}$

Defn:- (continuous):-

A function 'f' is continuous at 'a' if

$\lim_{x\to a} f(x) = f(a).$

Problems:-

① Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x\to 2} \left[3f(x) + f(x).g(x)\right] = 36.$

Find $f(2)$

Soln:-

Since f and g are continuous at 2,

$$\lim_{x \to 2} f(x) = f(2)$$

and

$$\lim_{x \to 2} g(x) = g(2) = 6.$$

Given that

$$\lim_{x \to 2} \left[3 f(x) + f(x) . g(x) \right] = 36.$$

$$\therefore \ 3 . f(2) + f(2) . g(2) = 36.$$

$$3 . f(2) + 6 . f(2) = 36.$$

$$9 . f(2) = 36.$$

$$f(2) = 4$$

Derivatives:-

Defn (Derivative of a function):-

The derivative of a function 'f' at a number 'a' is defined by $f'(a)$ and

$$f'(a) = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right]$$ provided this limit exists.

Formulas

Eqn of Tangent Line :-

The eqn of the tangent line of the curve $y = f(x)$ at a point $(a, f(a))$ is given by

$$\boxed{y - f(a) = m(x - a)}$$ where $m = f'(a)$.

'm' means Slope of tangent.

Eqn of Normal Line :-

The eqn of normal line of the curve $y = f(x)$ at a point $(a, f(a))$ is given by

$$\boxed{y - f(a) = \frac{-1}{m}(x - a)}$$ where $m = f'(a)$.

m → slope of tangent.

Problems :-

① Find the slope of the tangent of the parabola $y = 4x - x^2$ at $(1, 3)$.

Soln.

$y = 4x - x^2$.

$y = f(x)$

$\therefore f(x) = 4x - x^2$.

$f'(x) = 4 - 2x$.

$f'(1) = 4 - 2$

$m = f'(1). = 2$.

$\therefore$ Slope of tangent $m = 2$.

② Find an eqn of tangent line to the curve $y = \sqrt{x}$ at the point (1,1).

Soln:-

$$y = \sqrt{x}.$$

$$f(x) = y.$$

$$\therefore f(x) = \sqrt{x}.$$

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

$$m = f'(1) = \frac{1}{2\sqrt{1}} = 1/2.$$

$$\boxed{m = 1/2}$$

$$(1,1) \longrightarrow (a, f(a)).$$

$$a = 1.$$

$$f(a) = 1.$$

Eqn of tangent line

$$y - f(a) = m(x - a).$$

$$y - 1 = \frac{1}{2}(x - 1)$$

$$2y - 2 = x - 1.$$

$$x - 2y - 1 + 2 = 0.$$

$\boxed{x - 2y + 1 = 0.}$ is required tangent line.

(ii) Eqn of Normal line

$$y - f(a) = -\frac{1}{m}\left[x - a\right]$$

$$y - 1 = \frac{-1}{1/4}(x-1)$$

$$y - 1 = -4(x-1)$$

$$y - 1 = -4x + 4.$$

$$4x + y - 1 - 4 = 0.$$

$\boxed{4x + y - 5 = 0}$ is required normal line.

Maxima and Minima of Function of One Variable

Working Rule :-

1) Take given function as $f(x)$

2) Find $f'(x)$

3) Find roots of $f'(x) = 0$

Roots are called critical points.

4) Find $f''(x)$ at each of critical point 'a'.

(i) If $f''(a) < 0$. then $f(x)$ has maximum at $x = a$

(ii) If $f''(a) > 0$ then $f(x)$ has minimum at $x = a$.

(iii) If $f''(a) = 0$ ~~inconclusive~~ ~~follows~~.

then $f(x)$ has neither maximum nor minimum at $x = a$.

③ Find the eqn of tangent line and normal line to the curve $y = \sqrt[4]{x}$ at $(1,1)$.

Soln

$$y = \sqrt[4]{x} = x^{1/4}.$$

$$y = f(x).$$

$$\therefore f(x) = x^{1/4}.$$

$$f'(x) = \frac{1}{4} x^{1/4 - 1}$$

$$= \frac{1}{4} x^{-3/4}$$

$$f'(x) = \frac{1}{4x^{3/4}}.$$

$$f'(1) = \frac{1}{4(1)^{3/4}}.$$

$$m = f'(1) = 1/4.$$

$$(1,1) \longrightarrow (a, f(a))$$

$a = 1.$

$f(a) = 1.$

(i) Eqn of tangent line

$$y - f(a) = m[x - a]$$

$$y - 1 = \frac{1}{4}[x - 1]$$

$$4y - 4 = x - 1.$$

$\boxed{x - 4y + 3 = 0}$ is required tangent line.

Problems

① Discuss the maxima and minima of the function $x^5 - 5x^4 + 5x^3 + 10$.

Soln

$$f(x) = x^5 - 5x^4 + 5x^3 + 10.$$

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

$$f''(x) = 20x^3 - 60x^2 + 30x.$$

$$f'(x) = 0$$

$$\Rightarrow \quad 5x^4 + 15x^2 - 20x^3 = 0.$$

$$5x^2\left[\cancel{5}x^2 - 4x + 3\right] = 0.$$

$$x^2(x^2 - 4x + 3) = 0.$$

$$x^2(x-1)(x-3) = 0.$$

$$\boxed{x = 0, 1, 3}$$

∴ Critical points are 0, 1, 3.

	At $x=0$	At $x=1$	At $x=3$
$f''(x) = 20x^3 - 60x^2 + 30x$	0	$-10 < 0$	$90 > 0$
conclusion	neither max. nor min	maximum	minimum.

$\therefore$ $f(x)$ has maximum at $x = 1$

Maximum Value of $f(x) = (1)^5 - 5(1)^4 + 5(1) + 10$

$= 1 - 5 + 5 + 10$

$= 11.$

$f(x)$ has minimum value at $x = 3.$

Minimum Value of $f(x) = 3^5 - 5(3)^4 + 5(3) + 10$

$= -17.$

② Discuss the maxima and minima of the function $x\log x$.

Soln:-

$f(x) = x\log x.$

$f'(x) = x\left(\frac{1}{x}\right) + \log x(1)$

$= 1 + \log x.$

$f''(x) = \frac{1}{x}.$

$f'(x) = 0.$

$\Rightarrow$ $1 + \log x = 0.$

$\log x = -1.$

$e^{\log x} = e^{-1}$

$\boxed{x = \frac{1}{e}}$ is only critical point.

$f''\left(\frac{1}{e}\right) = \frac{1}{1/e} = e \simeq 2.53 > 0.$

$\therefore \frac{1}{e}$ is minimum point.

Minimum Value of 'x log x' $= \frac{1}{e} \log \frac{1}{e}$

$= \frac{1}{e} \log e^{-1}$

$= -1 \times \frac{1}{e}$

$= -1/e.$

Formula :-

Velocity $= \frac{ds}{dt}$

Acceleration $= \frac{d^2s}{dt^2}.$

Problem :-

① The eqn of motion of a particle is $S = t^3 - 3t$ where 'S' is in meters and 't' is in seconds.

Find ⓐ the velocity and acceleration as function of t

ⓑ the acceleration after 2 seconds.

ⓒ the acceleration when velocity is zero.

Soln:-

Given $S = t^3 - 3t$

ⓐ Velocity $= \frac{ds}{dt} = \frac{d}{dt}(t^3 - 3t)$

$= 3t^2 - 3.$

Acceleration = $\frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right)$

$= \frac{d}{dt}\left(3t^2 - 3\right)$

$= 6t.$

ⓑ Acceleration after 2 seconds

$= \left(\frac{d^2s}{dt^2}\right)_{t=2}$

$= \left[6t\right]_{t=2}$

$= 6 \times 2$

$= 12 \ m/s^2.$

ⓒ When velocity is zero

$3t^2 - 3 = 0$

$3t^2 = 3$

$t^2 = 1$

$\boxed{t = 1}$

Acceleration when velocity is zero

$= \left(\frac{d^2s}{dt^2}\right)_{t=1}$

$= \left[6t\right]_{t=1}$

$= 6 m/s^2.$

Derivatives using First Principle Rule :-

① For any rational number n, prove that $\lim\limits_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ $(a \neq 0)$

Soln :-

Case (i) Let n be a +ve integer.

Sub a+h for x and we get

$$\frac{x^n - a^n}{x - a} = \frac{(a+h)^n - a^n}{(a+h) - a} = \frac{(a+h)^n - a^n}{h}.$$

Expanding the numerator by Binomial Theorem, we get

$$\frac{x^n - a^n}{x - a} = \frac{a^n + n.a^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + \dots + h^n - a^n}{h}$$

$$= na^{n-1} + \frac{(n)(n-1)}{2!}a^{n-2}h + \dots + h^{n-1}$$

All the other terms except the first term contain h.

As $x \to a$, $h \to 0$.

$$\therefore \lim_{x\to a} \frac{x^n - a^n}{x - a} = n.a^{n-1}.$$

Case (ii) Let $n = \frac{p}{q}$ where p and q are positive integers

Let $x = y^q$ and $a = b^q$.

Then as $x \to a$, $y^q \to b^q$.

So as $x \to a$, $y \to b$.

$$\frac{x^n - a^n}{x - a} = \frac{x^{p/q} - a^{p/q}}{x - a} = \frac{(y^q)^{p/q} - (b^q)^{p/q}}{y^q - b^p} = \frac{y^p - b^p}{y^q - b^q}$$

$$\therefore \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{y \to b} \frac{y^p - b^p}{y^q - b^q}$$

$$= \lim_{y \to b} \left[\left(\frac{y^p - b^p}{y - b} \right) \times \left(\frac{y - b}{y^q - b^q} \right) \right]$$

$$= \frac{\lim_{y \to b} \left(\frac{y^p - b^p}{y - b} \right)}{\lim_{y \to b} \left(\frac{y^q - b^q}{y - b} \right)}$$

$$= \frac{p . b^{p-1}}{q . b^{q-1}}$$

$$= n . a^{n-1}$$

Case(iii)

$n = -m$ where m is a +ve integer or fraction.

$$\frac{x^n - a^n}{x - a} = \frac{x^{-m} - a^{-m}}{x - a} = \frac{a^m - x^m}{x^m a^m (x - a)}$$

$$= -\frac{1}{x^m a^m} \left(\frac{x^m - a^m}{x - a} \right)$$

$$\therefore \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = \lim_{x \to a} \left(-\frac{1}{x^m a^m} \cdot \frac{x^m - a^m}{x - a} \right)$$

$$= -\frac{1}{a^m . a^m} \quad m . a^{m-1} \quad \text{(by cases (i) \& (ii))}$$

$$= -m . a^{-m-1}$$

$$= n . a^{n-1} . \quad (\because n = -m).$$

Hence $\lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = n . a^{n-1}$ for all rational values of n.

② Find the derivative of x^n, where n is a rational number, using first principle rule. ③

Soln.

First Principle Rule.

$$\boxed{\begin{array}{l} \text{If} \quad y = f(x) \\ \dfrac{dy}{dx} = f'(x) = \lim\limits_{\Delta x \to 0} \left[\dfrac{f(x+\Delta x) - f(x)}{\Delta x} \right] \end{array}}$$

Take $f(x) = x^n = y$.

$$\therefore \frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^n - x^n}{\Delta x} \right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x} \right]$$

Formula $\boxed{\lim\limits_{x \to a} \left[\dfrac{x^n - a^n}{x - a} \right] = n . a^{n-1}}$

$$\therefore \frac{dy}{dx} = n . x^{n-1}.$$

③ Find the derivative of $\sin x$ using first principle rule.

Soln

$f(x) = \sin x$, $f(x+\Delta x) = \sin(x+\Delta x)$

$$\therefore \text{Derivative of } \sin x = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\sin(x+\Delta x) - \sin x}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\sin\left(x + \frac{\Delta x}{2} + \frac{\Delta x}{2}\right) - \sin\left(\frac{x + \Delta x}{2} - \frac{\Delta x}{2}\right)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\sin(A+B) - \sin(A-B)}{\Delta x} \right]$$ where $A = x + \frac{\Delta x}{2}$

$B = \frac{\Delta x}{2}.$

$$\boxed{\sin(A+B) - \sin(A-B) = 2\sin B \cos A}$$

$\therefore$ Derivative of $\sin x = \lim_{\Delta x \to 0} \left[\frac{2\sin B \cos A}{\Delta x} \right]$

$$= \lim_{\Delta x \to 0} \left[\frac{2\sin\frac{\Delta x}{2} \cdot \cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\sin\frac{\Delta x}{2}}{\Delta x/2} \right] \cdot \lim_{\Delta x \to 0} \left[\cos\left(x + \frac{\Delta x}{2}\right) \right]$$

$$= 1 \times \cos x$$

$$= \cos x \qquad \left(\because \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \right) = 1 \right).$$

④ Find the derivative of cos x using first principal rule

Soln. $f(x) = \cos x$, $f(x + \Delta x) = \cos(x + \Delta x)$.

Derivative of $\cos x = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$

$$= \lim_{\Delta x \to 0} \left[\frac{\cos(x + \Delta x) - \cos x}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\cos\left(x + \frac{\Delta x}{2} + \frac{\Delta x}{2}\right) - \cos\left(x + \frac{\Delta x}{2} - \frac{\Delta x}{2}\right)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\cos(A+B) - \cos(A-B)}{\Delta x} \right] \quad \text{where} \quad A = x + \frac{\Delta x}{2}, \quad B = \frac{\Delta x}{2}.$$

$$= \lim_{\Delta x \to 0} \left[\frac{-2\sin A \sin B}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{-2\sin\left(x + \frac{\Delta x}{2}\right) . \sin\left(\frac{\Delta x}{2}\right)}{\Delta x} \right]$$

$$= -\left[\lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right) \times \lim_{\Delta x \to 0} \left(\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \right) \right]$$

$$= -\left[\sin x \times 1 \right] \quad \left(\because \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \right) = 1 \right)$$

$$= -\sin x.$$

∴ Derivative of $\cos x = -\sin x$.

⑤ Find the derivative of $\log_a x$, using first principal rule.

<u>Soln</u> $f(x) = \log_a x$, $f(x+\Delta x) = \log_a (x+\Delta x)$.

Derivative of $\log_a x = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$

$$= \lim_{\Delta x \to 0} \left(\frac{\log_a (x+\Delta x) - \log_a x}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \left[\frac{\log_a \left(\frac{x+\Delta x}{x} \right)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\log_a \left(1 + \frac{\Delta x}{x}\right)}{\Delta x} \right] \longrightarrow ①$$

Put $\frac{\Delta x}{x} = h$.

Then $\frac{1}{\Delta x} = \frac{1}{xh}$.

Also $h \to 0$ as $\Delta x \to 0$.

① becomes

Derivative of $\log_a x = \lim_{h \to 0} \left[\frac{\log_a (1+h)}{xh} \right]$

$$= \frac{1}{x} \cdot \lim_{h \to 0} \left[\frac{\log_a (1+h)}{h} \right] \longrightarrow ②$$

Formula $\boxed{\lim_{h \to 0} \left[\frac{\log_a (1+h)}{h} \right] = \log_a e}$

$\therefore$ ② becomes

Derivative of $\log_a x = \frac{1}{x} \cdot \log_a e$.

⑥ Derive the derivative of product rule.

Soln:-

Let $y = uv$ where, u & v are functions of x.

Let Δy, Δu, Δv be the increments in y, u, v respectively corresponding to an increment Δx in x.

Then $$(y+\Delta y) = (u+\Delta u)\cdot(v+\Delta v).$$

$$\Delta y = (u+\Delta u)\cdot(v+\Delta v) - y$$

$$= (u+\Delta u)\cdot(v+\Delta v) - uv$$

$$= u\Delta v + v.\Delta u + \Delta u.\Delta v$$

$$\therefore \frac{\Delta y}{\Delta x} = u.\frac{\Delta v}{\Delta x} + v.\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}.\Delta v \longrightarrow ①$$

When $\Delta x \to 0$, $\Delta u \to 0$

$\Delta v \to 0$

$\Delta y \to 0$.

$$\therefore \frac{\Delta y}{\Delta x} \to \frac{dy}{dx}, \quad \frac{\Delta u}{\Delta x} \to \frac{du}{dx}, \quad \frac{\Delta v}{\Delta x} \to \frac{dv}{dx}.$$

$$\therefore \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = u.\frac{dv}{dx} + v.\frac{du}{dx} + 0$$

$$\therefore \frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}.$$

(or) $$\frac{d(uv)}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}.$$

⑦ Derive the derivative of Quotient rule.

Soln:- Let $y = \frac{u}{v}$ where u & v are functions of x.

When x becomes $x+\Delta x$, $y \to y+\Delta y$
$u \to u+\Delta u$
$v \to v+\Delta v$.

$$\boxed{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}}$$

$$\therefore \; y+\Delta y = \frac{u+\Delta u}{v+\Delta v}$$

$$\Delta y = \frac{u+\Delta u}{v+\Delta v} - y$$

$$= \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}. \quad \left(\because y = \frac{u}{v}\right)$$

$$= \frac{uv + v.\Delta u - uv - u\Delta v}{v(v+\Delta v)}$$

$$= \frac{v\Delta u - u\Delta v}{v^2 + \Delta v.v}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{v.\frac{\Delta u}{\Delta x} - u.\frac{\Delta v}{\Delta x}}{v^2 + v.\Delta v.}$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{\lim_{\Delta x \to 0}\left(v.\frac{\Delta u}{\Delta x} - u.\frac{\Delta v}{\Delta x}\right)}{\lim_{\Delta x \to 0}\left(v^2 + v.\Delta v\right)}$$

$$= \frac{v.\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} - u.\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x}}{v^2 + v.\lim_{\Delta x \to 0}\Delta v}$$

$$\frac{dy}{dx.} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

Derivatives of Elementary Functions:-

Important Formulas

1) $\frac{d}{dx}(x^n) = n.x^{n-1}$. 2) $\frac{d}{dx}(\log x) = \frac{1}{x}$ 3) $\frac{d}{dx}(e^x) = e^x$

4) $\frac{d}{dx}(e^{ax}) = a.e^{ax}$ 5) $\frac{d}{dx}(\sin x) = \cos x$ 6) $\frac{d}{dx}(\cos x) = -\sin x$

7) $\frac{d}{dx}(\tan x) = \sec^2 x$ 8) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ 9) $\frac{d}{dx}(\sec x) = \sec x . \tan$

~~8~~ 10) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x$.

11) $\frac{d}{dx}(uv) = u.v' + u'v$

12) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.u' - u.v'}{v^2}$

13) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

Problems

① Fend $\frac{dy}{dx}$ for the following functions.

(i) $y = 2x^4 - 3x^3 + 12x^2 + 5$.

Soln:-

$$\frac{dy}{dx} = 8x^3 - 9x^2 + 24x$$

(ii) $y = e^{-x} + \log x$.

Soln:- $\frac{dy}{dx} = -e^{-x} + \frac{1}{x} = \frac{1 - x.e^{-x}}{x.}$

(iii) $y = \frac{x^3 - 2x^2 + 5}{x^2}$

Soln

$$y = \frac{x^3 - 2x^2 + 5}{x^2} = \frac{u}{v.}$$

$$\therefore \frac{dy}{dx} = \frac{x^2(3x^2 - 4x) - (x^3 - 2x^2 + 5)(2x)}{x^4}$$

$$\frac{dy}{dx} = \frac{3x^4 - \cancel{4x^3} - 2x^4 + \cancel{4x^3} - 10x}{x^4} = \frac{x^4 - 10x}{x^4} = \frac{x^3 - 10}{x^3}$$

(iv) $y = e^x + 3\tan x + \log x^4$

Soln

$y = e^x + 3\tan x + 4\log x.$

$\therefore \frac{dy}{dx} = e^x + 3\sec^2 x + \frac{4}{x}.$

(v) $y = \sin 3 + 2\sec x.$

Soln

$y = \sin 3 + 2\sec x.$

$\frac{dy}{dx} = 0 + 2\sec x . \tan x.$ $\left(\because \sin 3 \text{ is constant, } \frac{d}{dx}(\sin 3) = 0\right)$

$\frac{dy}{dx} = 2\sec x . \tan x.$

2) Find $\frac{dy}{dx}$ for the following functions.

(i) $y = (4x^2-3)(2x+1).$

Soln:- $\frac{dy}{dx} = (4x^2-3)(2) + (2x+1)(8x)$

$= 8x^2 - 6 + 16x^2 + 8x$

$\frac{dy}{dx} = 24x^2 + 8x - 6.$

(ii) $y = (x^2+7x+2)(e^x - \log x)$

Soln:-

$\frac{dy}{dx} = (x^2+7x+2)\left(e^x - \frac{1}{x}\right) + (e^x - \log x)(2x+7)$

(iii) $y = e^x \sin x.$

Soln:- $\frac{dy}{dx} = e^x(\cos x) + \sin x . e^x$

$\frac{dy}{dx} = e^x(\sin x + \cos x).$

(iv) $y = (3\sec x - 4\operatorname{cosec} x)(2\sin x + 5\cos x)$

Soln:-

$\frac{dy}{dx} = \left[3\sec x \tan x - 4(-\operatorname{cosec} x . \cot x)\right] . (2\sin x + 5\cos x)$

$+ (3\sec x - 4\operatorname{cosec} x)(2\cos x + 5(-\sin x))$

$$\frac{dy}{dx} = \left(3\sec x.\tan x + 4\,\mathrm{cosec}\,x.\cot x\right)\left(2\sin x + 5\cos x\right)$$
$$+ \left(3\sec x - 4\,\mathrm{cosec}\,x\right)\left(2\cos x - 5\sin x\right)$$

(v) $y = x^2 e^x \sin x.$

Soln:-

$$\frac{dy}{dx} = x^2 e^x \left(\cos x\right) + x^2 \sin x \left(e^x\right) + e^x \sin x \left(2x\right).$$
$$= x \cdot e^x \left(x.\cos x + x\sin x + 2\sin x\right)$$

③ Find $\frac{dy}{dx}$ for the following functions.

Soln:-

(i) $y = \frac{2x+3}{3x-5}.$

Soln:-

$$\frac{dy}{dx} = \frac{(3x-5)(2) - (2x+3)(3)}{(3x-5)^2}$$
$$= \frac{6x - 10 - (6x+9)}{(3x-5)^2}$$
$$\frac{dy}{dx.} = \frac{-19}{(3x-5)^2}$$

(ii) $y = \frac{\tan x + 1}{\tan x - 1}$

Soln:-

$$\frac{dy}{dx} = \frac{(\tan x - 1)(\sec^2 x) - (\tan x + 1).(\sec^2 x)}{(\tan x - 1)^2}$$
$$= \frac{\tan x.\sec^2 x - \sec^2 x - \tan x \sec^2 x - \sec^2 x}{\tan^2 x - 2\tan x + 1}$$
$$\frac{dy}{dx.} = \frac{-2\sec^2 x}{\sec^2 x - 2\tan x.}$$

(iii) $y = \dfrac{\cos x + \log x}{x^2 + e^x}$

Soln:-

$$\frac{dy}{dx} = \frac{(x^2+e^x)(-\sin x+\frac{1}{x}) - (\cos x+\log x)(2x+e^x)}{(x^2+e^x)^2}$$

(iv) $y = \dfrac{x^2+e^x \sin x}{\cos x + \log x}$.

Soln:-

$$\frac{dy}{dx} = \frac{(\cos x+\log x)(2x+e^x\sin x+e^x\cos x) - (x^2+e^x\sin x)(-\sin x+1/x}{(\cos x+\log x)^2}.$$

(v) $y = \dfrac{\sin x+\cos x}{\sin x-\cos x}$.

Soln:-

$$\frac{dy}{dx} = \frac{(\sin x-\cos x)(\cos x-\sin x) - (\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$= \frac{-(\sin x-\cos x)^2 - (\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$= -\left[\frac{(\sin x-\cos x)^2+(\sin x+\cos x)^2}{(\sin x-\cos x)^2}\right]$$

$$= -\left[\frac{\sin^2x+\cos^2x-2\sin x\cos x+\sin^2x+\cos^2x+2\sin x\cos x}{(\sin x-\cos x)^2}\right]$$

$$= -\left[\frac{2}{\sin^2x+\cos^2x-2\sin x\cos x}\right]$$

$$\frac{dy}{dx}. = -\left[\frac{2}{1-2\sin x\cos x}\right]$$

(v) $y = \tan(\log x)$.

Soln :-

$$\frac{dy}{dx} = \sec^2(\log x) \times \frac{d}{dx}(\log x)$$

$$= \sec^2(\log x) \times \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{\sec^2(\log x)}{x}.$$

(vi) $y = e^{\sin(\log x)}$.

Soln:-

$$\frac{dy}{dx} = e^{\sin(\log x)} \times \frac{d}{dx}(\sin(\log x))$$

$$= e^{\sin(\log x)} \times \cos(\log x) \times \frac{d}{dx}(\log x)$$

$$= e^{\sin(\log x)} \times \cos(\log x) \times \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{e^{\sin(\log x)} \times \cos(\log x)}{x}.$$

(vii) $y = \sqrt{1+\cot x}$

Soln:-

$$\frac{dy}{dx} = \frac{1}{2}(1+\cot x) \times (-\operatorname{cosec}^2 x)$$

$$\frac{dy}{dx} = \frac{-(1+\cot x)\operatorname{cosec}^2 x}{2}.$$

(or)

$$\frac{dy}{dx} = -\frac{(1+\cot x)(1+\cot^2 x)}{2}.$$ $\left(\because \operatorname{cosec}^2 x = 1+\cot^2 x\right)$.

⑤ Find $\frac{dy}{dx}$ for the following functions.

Soln:-

(i) $x = at^2$, $y = 2at$.

$$\boxed{\text{Formula } \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}.}$$

④ Find $\frac{dy}{dx}$ for the following functions.

(i) $y = \sin(x^2+2x+3)$

Soln:-

$$\frac{dy}{dx} = \cos(x^2+2x+3) \times (2x+2)$$

$$= (2x+2) \cdot \cos(x^2+2x+3).$$

(ii) $y = \log\sqrt{x}$.

Soln:-

$$y = \log\sqrt{x}$$

$$y = \log x^{1/2}$$

$$y = \frac{1}{2}\log x.$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{2x}.$$

(iii) $y = e^{\sin x}$

Soln:-

$$\frac{dy}{dx} = e^{\sin x} \cdot \frac{d}{dx}(\sin x)$$

$$= e^{\sin x} \cdot \cos x$$

$$\frac{dy}{dx.} = \cos x \cdot e^{\sin x}.$$

(iv) $y = e^{\sin x^2}$.

Soln:-

$$\frac{dy}{dx} = e^{\sin x^2} \frac{d}{dx}(\sin x^2)$$

$$= e^{\sin x^2} \cos x^2 \cdot \frac{d}{dx}(x^2)$$

$$= e^{\sin x^2} \times \cos x^2 \times 2x$$

$$\frac{dy}{dx.} = 2x \cdot e^{\sin x^2} \cdot \cos x^2.$$

(v) $y = \tan(\log x)$.

Soln :-

$$\frac{dy}{dx} = \sec^2(\log x) \times \frac{d}{dx}(\log x)$$

$$= \sec^2(\log x) \times \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{\sec^2(\log x)}{x}$$

(vi) $y = e^{\sin(\log x)}$

Soln :-

$$\frac{dy}{dx} = e^{\sin(\log x)} \times \frac{d}{dx}(\sin(\log x))$$

$$= e^{\sin(\log x)} \times \cos(\log x) \times \frac{d}{dx}(\log x)$$

$$= e^{\sin(\log x)} \times \cos(\log x) \times \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{e^{\sin(\log x)} \times \cos(\log x)}{x}.$$

(vii) $y = \sqrt{1+\cot x}$

Soln :-

$$\frac{dy}{dx} = \frac{1}{2}(1+\cot x) \times (-\operatorname{cosec}^2 x)$$

$$\frac{dy}{dx} = \frac{-(1+\cot x)\operatorname{cosec}^2 x}{2}.$$

(or)

$$\frac{dy}{dx} = -\frac{(1+\cot x)(1+\cot^2 x)}{2}. \quad (\because \operatorname{cosec}^2 x = 1+\cot^2 x).$$

⑤ Find $\frac{dy}{dx}$ for the following functions.

Soln :-

(i) $x = at^2$, $y = 2at$.

Formula $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$.

⑥ Find $\frac{dy}{dx}$ for the following functions.

(1) $x^3 + 8xy + y^3 = 64$.

Soln:-

Given $x^3 + 8xy + y^3 = 64$.

Diff. w.r.t x.

$$3x^2 + 8\left(x.\frac{dy}{dx} + y(1)\right) + 3y^2\frac{dy}{dx} = 0.$$

$$3x^2 + 8x.\frac{dy}{dx} + 8y + 3y^2\frac{dy}{dx} = 0.$$

$$(8x + 3y^2)\frac{dy}{dx} = -(3x^2 + 8y)$$

$$\therefore \frac{dy}{dx} = -\frac{(3x^2 + 8y)}{(8x + 3y^2)}$$

(ii) $x^3 + y^3 = 3axy$.

Soln:-

$$3x^2 + 3y^2.\frac{dy}{dx} = 3a\left(x.\frac{dy}{dx} + y(1)\right)$$

$\div 3$, $$x^2 + y^2.\frac{dy}{dx} = ax.\frac{dy}{dx} + ay.$$

$$y^2.\frac{dy}{dx} - ax.\frac{dy}{dx} = ay - x^2.$$

$$\frac{dy}{dx}(y^2 - ax) = ay - x^2.$$

$$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

(iii) $e^x + e^y = e^{x+y}$

Soln.

Diff w.r.t 'x'

$$e^x + e^y.\frac{dy}{dx} = e^{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$e^x + e^y.\frac{dy}{dx} = e^{x+y} + e^{x+y}.\frac{dy}{dx}.$$

$$\therefore e^y.\frac{dy}{dx} - e^{x+y}.\frac{dy}{dx} = e^{x+y} - e^x.$$

$$\frac{dy}{dx} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}}$$

(iv) $(1+y^2)\sec x - y\cot x + 1 = x^2$

Soln Diff. w.r.t 'x'

$$(1+y^2)\sec x.\tan x + \sec x\left(0+2y.\frac{dy}{dx}\right) - \left[y(-\operatorname{cosec}^2 x) + \cot x.\frac{dy}{dx}\right]$$

$$+0 = 2x.$$

$$(1+y^2)\sec x.\tan x + 2y.\sec x.\frac{dy}{dx} + y\operatorname{cosec}^2 x - \cot x.\frac{dy}{dx} = 2x$$

$$(2y\sec x - \cot x)\frac{dy}{dx} = 2x - (1+y^2)\sec x.\tan x - y\operatorname{cosec}^2 x.$$

$$\therefore \frac{dy}{dx} = \frac{2x - (1+y^2)\sec x.\tan x - y\operatorname{cosec}^2 x}{(2y\sec x - \cot x)}$$

(v) $ax^2+by^2+2gx+2fy+2hxy+c=0.$

(Hent: a, b, g, f, h, c are constants)

Soln:-

Given $ax^2+by^2+2gx+2fy+2hxy+c=0.$

Diff. w.r.t 'x'

$$2ax + 2by.\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} + 2h\left(x.\frac{dy}{dx} + y(1)\right) + 0 = 0$$

÷ 2, $$ax + by\frac{dy}{dx} + g + f.\frac{dy}{dx} + xh.\frac{dy}{dx} + hy = 0$$

$$(ax+g+hy) + \frac{dy}{dx}(by+f+xh) = 0.$$

$$\therefore \frac{dy}{dx} = -\frac{(ax+g+hy)}{(by+f+xh)}$$

Derivative using Logarithmic

1) Find $\frac{dy}{dx}$, when $y = x^{\sin x}$.

Soln:-

$$y = x^{\sin x}.$$

Taking logarithm on both sides

$$\log y = \log x^{\sin x} = \sin x . \log x.$$

(ie) $\log y = \sin x . \log x.$

Diff. both sides w.r.t 'x'

$$\frac{1}{y} . \frac{dy}{dx} = \sin x \left(\frac{1}{x}\right) + \log x \; (+\cos x)$$

$$\frac{1}{y} . \frac{dy}{dx} = \frac{\sin x}{x} + \cos x . \log x .$$

$$\therefore \frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x . \log x\right]$$

Since $y = x^{\sin x}$, $\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x . \log x\right]$

2) Find $\frac{dy}{dx}$, when $y = (\tan x)^{\log x}$.

Soln:- $y = (\tan x)^{\log x}.$

Taking logarithm on both sides

$$\log y = \log (\tan x)^{\log x}$$

$$\log y . = \log x . \log(\tan x)$$

Diff. w.r.t 'x'

$$\frac{1}{y} . \frac{dy}{dx} = \log x\left(\frac{1}{\tan x} \times \sec^2 x\right) + \log(\tan x) \times \frac{1}{x}$$

$$\frac{1}{y} . \frac{dy}{dx} = \log x \left(\cot x \times \sec^2 x\right) + \frac{\log(\tan x)}{x.}$$

$$= \log x \left(\frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x}\right) + \frac{\log(\tan x)}{x}$$

$$= \log x \left(\frac{1}{\sin x \cos x}\right) + \frac{\log(\tan x)}{x}$$

$$\frac{1}{y} . \frac{dy}{dx} = \frac{\log x}{\sin x . \cos x} + \frac{\log(\tan x)}{x.}$$

$$\frac{dy}{dx} = y\left[\frac{\log x}{\sin x . \cos x} + \frac{\log(\tan x)}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\tan x)^{\log x}\left[\frac{\log x}{\sin x \cos x} + \frac{\log(\tan x)}{x}\right]$$

3. Find $\frac{dy}{dx}$, when $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$.

Soln:-

$$y = (\tan x)^{\cot x} + (\cot x)^{\tan x}.$$

Take $u = (\tan x)^{\cot x}$, $v = (\cot x)^{\tan x}$.

Then $y = u + v$.

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \longrightarrow ①$$

Now $u = (\tan x)^{\cot x}$

$$\therefore \log u = \log (\tan x)^{\cot x} = \cot x \cdot \log(\tan x)$$

$$\Rightarrow \log u = \cot x \cdot \log(\tan x)$$

Diff w.r.t 'x'

$$\frac{1}{u} \cdot \frac{du}{dx} = \cot x \left[\frac{1}{\tan x} \times \sec^2 x\right] + \log(\tan x) \times -\csc^2 x.$$

$$= \sec^2 x - \csc^2 x \cdot \log(\tan x)$$

$$\therefore \frac{du}{dx} = u\left[\sec^2 x - \csc^2 x \cdot \log(\tan x)\right]$$

$$\frac{du}{dx} = \tan x^{(\cot x)} \left[\sec^2 x - \csc^2 x \cdot \log(\tan x)\right] \longrightarrow ②$$

Now $v = (\cot x)^{\tan x}$

$$\log v = \log(\cot x)^{\tan x}$$

$$\log v = \tan x \cdot \log(\cot x)$$

Diff w.r.t x.

$$\frac{1}{v} \cdot \frac{dv}{dx} = \tan x \left(\frac{1}{\cot x} \times -\csc^2 x\right) + \log(\cot x) \times \sec^2 x$$

$$= -\csc^2 x + \sec^2 x \cdot \log(\cot x)$$

$$\therefore \frac{dv}{dx} = v\left[-\csc^2 x + \sec^2 x \cdot \log(\cot x)\right]$$

$$\frac{dv}{dx} = (\cot x)^{\tan x}\left[-\csc^2 x + \sec^2 x \cdot \log(\cot x)\right] \longrightarrow ③$$

Sub ② & ③ in ①

$$\frac{dy}{dx} = (\tan x)^{\cot x}\left[\sec^2 x - \csc^2 x \cdot \log(\tan x)\right] + (\cot x)^{\tan x}\left[-\csc^2 x + \sec^2 x \log(\cot\right]$$

4) Find $\frac{dy}{dx}$, when $y = \frac{(1-x)\sqrt{x^2+2}}{(x+3)\sqrt{x-1}}$.

Soln:-

$$y = \frac{(1-x)\,(x^2+2)^{1/2}}{(x+3)\,(x-1)^{1/2}}$$

Taking logarithm on both sides.

$$\log y = \log\left[\frac{(1-x)\,(x^2+2)^{1/2}}{(x+3)\,(x-1)^{1/2}}\right]$$

$$= \log\left[(1-x)\,(x^2+2)^{1/2}\right] - \log\left[(x+3)\,(x-1)^{1/2}\right] \qquad \left(\because \log\left(\frac{A}{B}\right) = \log A - \log B\right)$$

$$= \log(1-x) + \log(x^2+2)^{1/2} - \log(x+3) - \log(x-1)^{1/2} \qquad \left(\because \log(AB) = \log A + \log B\right)$$

$$\log y = \log(1-x) + \frac{1}{2}\log(x^2+2) - \log(x+3) - \frac{1}{2}\log(x-1) \qquad \left(\because \log A^m = m\log A\right)$$

Diff both sides w.r.t 'x'

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{(1-x)}\times(-1) + \frac{1}{2}\left(\frac{1}{x^2+2}\right)\times 2x - \frac{1}{(x+3)}(1) - \frac{1}{2}\times\left(\frac{1}{x-1}\right)\times 1.$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{-1}{1-x} + \frac{x}{x^2+2} - \frac{1}{x+3} - \frac{1}{2(x-1)}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{-1}{1-x} + \frac{x}{x^2+2} - \frac{1}{x+3} - \frac{1}{2(x-1)}\right]$$

$$\frac{dy}{dx} = \frac{(1-x)\,(x^2+2)^{1/2}}{(x+3)\,(x-1)^{1/2}}\left[\frac{-1}{1-x} + \frac{x}{x^2+2} - \frac{1}{x+3} - \frac{1}{2(x-1)}\right]$$

5) Find $\frac{dy}{dx}$, when $y = (x^2+x+1)^{\sqrt{x-1}}$

Soln:-

$$y = (x^2+x+1)^{\sqrt{x-1}}$$

Taking logarithm on both sides

$$\log y = \log(x^2+x+1)^{\sqrt{x-1}}$$

$$\log y = \sqrt{x-1}\cdot\log(x^2+x+1)$$

$$\log y = (x-1)^{1/2}\log(x^2+x+1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = (x-1)^{1/2}\left[\frac{1}{x^2+x+1}\times(2x+1)\right]+\left(\log(x^2+x+1)\times\frac{1}{2}(x-1)^{1/2-1}\times 1\right)$$

$$= \frac{(x-1)^{1/2}(2x+1)}{x^2+x+1}+\frac{\log(x^2+x+1)}{2(x-1)^{1/2}}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{(x-1)^{1/2}(2x+1)}{x^2+x+1}+\frac{\log(x^2+x+1)}{2(x-1)^{1/2}}\right]$$

$$\frac{dy}{dx} = (x^2+x+1)^{\sqrt{x-1}}\left[\frac{(x-1)^{1/2}(2x+1)}{x^2+x+1}+\frac{\log(x^2+x+1)}{2(x-1)^{1/2}}\right]$$

6) Find $\frac{dy}{dx}$, when $y = \sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\dots\infty}}}$.

Soln:-

$$y = \sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\dots\infty}}}$$

$$y = \sqrt{\sin x + y}$$

Squaring both sides

$$y^2 = \sin x + y.$$

Diff w.r.t 'x'

$$2y.\frac{dy}{dx} = \cos x+\frac{dy}{dx}..$$

$$(2y-1)\frac{dy}{dx} = \cos x.$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1}.$$

Maxima and Minima of functions of one Variable

① Find the maximum and minimum values of $f(x) = x^4 - 3x^3 + 3x^2 - x$

Soln:-

$$f(x) = x^4 - 3x^3 + 3x^2 - x.$$

$$f'(x) = 4x^3 - 9x^2 + 6x - 1.$$

$$f''(x) = 12x^2 - 18x + 6.$$

Now $f'(x) = 0$ implies $4x^3 - 9x^2 + 6x - 1 = 0 \longrightarrow ①$

Clearly $x = 1$ is a root of ①

$$\begin{array}{r|rrrr} 1 & 4 & -9 & +6 & -1 \\ & 0 & 4 & -5 & 1 \\ \hline & 4 & -5 & +1 & \boxed{0} \end{array}$$

$\therefore \quad 4x^3 - 9x^2 + 6x - 1 = (x-1)(4x^2 - 5x + 1) = 0.$

$\Rightarrow (x-1)(4x-4)(4x-1) = 0$

$\Rightarrow x = 1, \; x = 1, \; x = 1/4.$

$f''(x) = 12x^2 - 18x + 6.$	
At $x = 1$ $f''(1) = 12 - 18 + 6 = 0$	$\therefore f(x)$ has neither maximum nor minimum at $x = 1$.
At $x = 1/4$. $f''(1/4) = 12(1/16) - 18(1/4) + 6$ $= \frac{3}{4} - \frac{9}{2} + 6 = \frac{3-18+24}{4}$ $= 9/4 > 0$	$\therefore f(x)$ has minimum value at $x = 1/4$. Minimum value of $f(x) = (1/4)^4 - 3(1/4)^3 + 3(1/4)^2 - 1/4$ $= -27/256$.

② Find the absolute maximum and absolute minimum values of $f(x) = 2x^3 - 3x^2 - 12x + 1$ on the interval $[-2, 3]$

Soln :-

Rule :-

(i) Find $f'(x)$

(ii) Find Roots of $f'(x) = 0$

(iii) Find $f(x)$ values for these roots and end pts of interval

(iv) Determine absolute maximum and absolute minimum.

$f(x) = 2x^3 - 3x^2 - 12x + 1.$

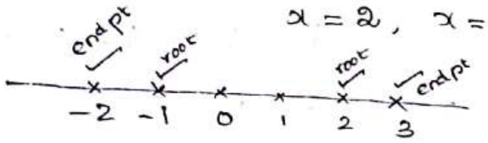
$f'(x) = 6x^2 - 6x - 12.$

$f'(x) = 0 \implies 6x^2 - 6x - 12 = 0$

$\div 6, \quad x^2 - x - 2 = 0.$

$(x-2)(x+1) = 0.$

$x = 2, \quad x = -1.$



$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1$

$= -16 - 12 + 24 + 1$

$= -3.$

$f(-1) = 2(-1) - 3(1) - 12(-1) + 1$

$= -2 - 3 + 12 + 1$

$= 8.$

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 16 - 12 - 24 + 1$

$= -19.$

$f(3) = 2(27) - 3(9) - 12(3) + 1$

$= 54 - 27 - 36 + 1$

$= -8.$

x	-2	-1	2	3
$f(x)$	-3	(max) 8	-19 (min)	-8

$\therefore f(-1) = 8$ is absolute maximum of $f(x)$

$f(2) = -19$ is absolute minimum of $f(x)$.

③ Find two numbers whose sum is 100 and whose product is maximum.

Soln:- Let x and y be required numbers.

Then $x+y=100 \Rightarrow y=100-x \longrightarrow ①$

Aim:- To find x and y such that xy is maximum.

Hint:- Decide the function as ~~whether~~ what you going to find maximum or minimum.

Take $f(x) = xy$

From ① $y = 100-x$

$\therefore f(x) = x(100-x) = 100x - x^2.$

$f'(x) = 100-2x.$

$f''(x) = -2.$

Now $f'(x)=0 \Rightarrow 100-2x=0$

$\Rightarrow \boxed{x=50}$

Put $x=50$ in $f''(x)$

(i) $f''(50) = -2<0.$

$\therefore f(x)$ is maximum at $x=50$.

Also from ①, $y=100-50$

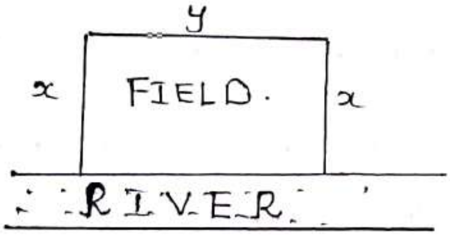
$\boxed{y=50}$

$\therefore$ $x=50$ & $y=50$ are required 2 numbers such tha

whose sum is 100 and product is maximum.

④ A farmer has 1200 m of fencing and want to fence of a rectangular field that borders a straight river. He needs no fence along the river. What are the dimension of the field that has the largest area?

Soln:-



Let length of field = y m

Breadth of field = x m.

The farmer needs no fence along the river.

$$\therefore \quad 2x + y = 1200 \implies y = 1200 - 2x \longrightarrow ①$$

Aim:- To find x and y such that "xy = Area of field" is maximum.

Take $f(x) = xy$.

∴ Our aim is to find x and y such that $f(x) = xy$ is maximum.

$$f(x) = xy = x(1200 - 2x) \qquad (\because \text{using } ①)$$

$$f(x). = 1200x - 2x^2$$

$$f'(x) = 1200\cancel{0} - 4x.$$

$$f''(x) = -4.$$

Now $f'(x) = 0 \implies 1200 - 4x = 0$

$$\implies 4x = 1200$$

$$\implies \boxed{x = 300}$$

Put $x = 300$ in $f''(x)$, $f''(300) = -4 < 0$.

∴ $f(x) = xy$ is maximum at $x = 300$.

From ① $y = 1200 - 2(300)$

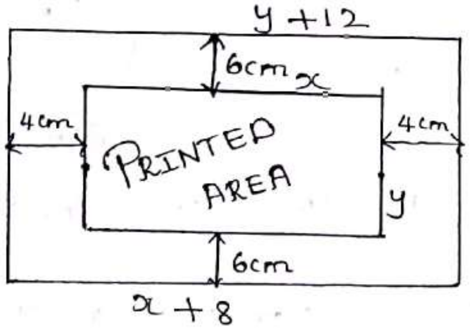
$$\boxed{y = 600}$$

Dimension of the field are 'x' meter, 'y' meter

∴ Dimension of the field are 300 meter, 600 meter.

⑤ The top and bottom margins of a poster are each 6cm and the side margins are each 4 cm. If the area of the printed material on the poster is fixed at 384 cm^2. Find the dimension of the poster with the smallest area.

Soln:-



Let x and y be length & breadth of PRINTED AREA.

Then length of poster $= x+8$

Breadth of Poster $= y+12$.

Aim:- To find x and y such that "the area of whole poster is minimum."

Area of whole poster $= (x+8)(y+12)$" is maximum

Given that $xy = 384 \implies y = \frac{384}{x} \rightarrow$ ①

Take $f(x) = (x+8)(y+12)$

$= xy + 8y + 12x + 96$

$= 384 + 8\left(\frac{384}{x}\right) + 12x + 96$ $\left(\because \text{from ①}, y = \frac{384}{x} \text{ \& } xy = 384\right)$

$f(x) = 480 + \frac{3072}{x} + 12x$

$f'(x) = \frac{-3072}{x^2} + 12.$

$f''(x) = \frac{6144}{x^3}.$

Now $f'(x) = 0 \implies \frac{-3072}{x^2} + 12 = 0.$

$\implies 12 = \frac{3072}{x^2}$

$\implies x^2 = \frac{3072}{12} = 256$

$\implies x = \pm 16$

$\implies \boxed{x = 16}$ ($\because x$ cannot be $-$ve)

Put $x=16$ in $f''(x)$, $f''(16) = \frac{6144}{16^3} > 0.$

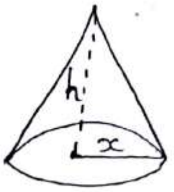
$\therefore f(x)$ is minimum at $x = 16$.

From ①, $y = \frac{384}{16} = 24.$ ∴ $\boxed{y = 24}$

∴ Dimension of the poster are $(x+8)$ cm, $(y+12)$ cm.

⑥ Show that the volume of the largest right circular cone that can be Inscribed in a sphere of radius 'a' is $\frac{8}{27} \times$ (volume of sphere).

soln

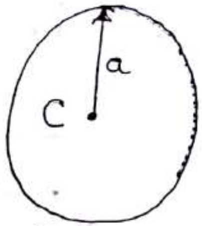


cone

$x \rightarrow$ radius

$h \rightarrow$ height

Volume $= \frac{1}{3}\pi x^2 h$.



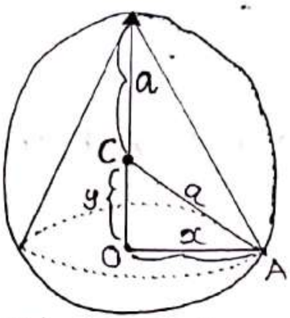
Sphere

radius = a

Centre = C

Volume $= \frac{4}{3}\pi a^3$

Now consider the Cone inside Sphere.

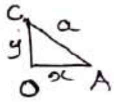


Now height of cone = a+y

radius of cone = x

radius of sphere = a.

When a cone inside a sphere, volume of cone $= \frac{1}{3}\pi x^2 (a+y)$. $\longrightarrow$ ①

From the triangle  we have $x^2+y^2 = a^2$.

$\Rightarrow x^2 = a^2 - y^2 \longrightarrow$ ②

Sub ② in ① Volume of cone $= \frac{1}{3}\pi (a^2-y^2)(a+y)$

$= \frac{1}{3}\pi (a^3 - ay^2 + a^2y - y^3)$

Aim:- Volume of largest circular cone $= \frac{8}{27} \times$ Volume of sphere

$\therefore$ Volume of cone must be maximum.

$\therefore$ Take $f(y) = \frac{1}{3}\pi (a^3 - ay^2 + a^2y - y^3)$

Diff. w.r.t 'y'

$f'(y) = \frac{1}{3}\pi (0 - 2ay + a^2 - 3y^2)$ [$\because$ a is const]

$f'(y) = \frac{1}{3}\pi (-2ay + a^2 - 3y^2)$.

$f''(y) = \frac{1}{3}\pi (-2a + 0 - 6y)$

$f''(y) = \frac{1}{3}\pi (-2a - 6y)$

Now $f'(y) = 0 \Longrightarrow \frac{1}{3}\pi(-2ay + a^2 - 3y^2) = 0$

$\Longrightarrow -2ay + a^2 - 3y^2 = 0$

$\Longrightarrow 3y^2 + 2ay - a^2 = 0$

$\Rightarrow (3y+3a)(3y-a) = 0 \Longrightarrow y = -a$ (or) $y = a/3$

$-3a^2$

$3a \mid -a$

$3y \mid 3y$

Since y (height) cannot be negative $\boxed{y = \frac{a}{3}}$

Put $y = a/3$ in $f''(y) = \frac{1}{3}\pi\left(-2a - 6\left(\frac{a}{3}\right)\right)$

$$= -\frac{4\pi a}{3} < 0.$$

$\therefore$ $f(y)$ is maximum at $y = a/3$.

(ω) Volume of cone is maximum at $y = a/3$.

$\therefore$ Volume of largest cone $= \frac{1}{3}\pi x^2 h$

From ② $x^2 = a^2 - a^2/9 = \frac{8a^2}{9}$.

$\therefore$ Volume of largest cone $= \frac{1}{3}\pi \times \frac{8a^2}{9} \times \left(a + \frac{a}{3}\right)$

$$= \frac{1}{3} \times \pi \times \frac{8a^2}{9} \times \left(\frac{4a}{3}\right)$$

$$= \frac{4}{3} \times \pi \times \frac{8}{27} \times a^3$$

$$= \frac{8}{27} \times \left(\frac{4}{3} \times \pi \times a^3\right)$$

Volume of largest cone $= \frac{8}{27} \times$ (volume of sphere).

Hence the proof.

× ——— ×.

Unit-III

Function of Several Variables.

Defn (Function of Two Variable) :-

A symbol 'z' which has a definite value for every pair of values x and y is called a function of two independent variables x and y and we write $z = f(x,y)$.

Defn :- (Limit)

The function $f(x,y)$ is said to tend to a limit 'L' as $x \to a$ and $y \to b$ iff

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = L.$$

Problems :-

① Evaluate $\lim_{\substack{x \to 1 \\ y \to 2}} \left(\frac{3x^2y}{x^2+y^2+5} \right)$.

Solution:-

$$\lim_{\substack{x \to 1 \\ y \to 2}} \left(\frac{3x^2y}{x^2+y^2+5} \right) = \lim_{x \to 1} \left(\lim_{y \to 2} \frac{3x^2y}{x^2+y^2+5} \right)$$

$$= \lim_{x \to 1} \left(\frac{6x^2}{x^2+4+5} \right)$$

$$= \frac{6}{1+9}$$

$$= 3/5.$$

② Evaluate $\lim\limits_{\substack{x \to \infty \\ y \to 2}} \left(\dfrac{xy+5}{x^2+2y^2} \right)$

Soln:-

$$\lim_{\substack{x \to \infty \\ y \to 2}} \left(\frac{xy+5}{x^2+2y^2} \right) = \lim_{x \to \infty} \left(\lim_{y \to 2} \frac{xy+5}{x^2+2y^2} \right)$$

$$= \lim_{x \to \infty} \left(\frac{2x+5}{x^2+8} \right)$$

$$= \lim_{x \to \infty} \left(\frac{x^2}{x^2} \left(\frac{2/x + 5/x^2}{1 + 8/x^2} \right) \right)$$

$$= \frac{0+0}{1+0}$$

$$= 0.$$

Partial Differentiation:-

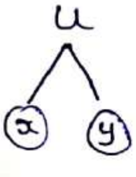
If z is a function of x and y then we can differentiate 'z' partially w.r.t x and y.

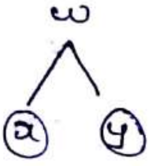
$\dfrac{\partial z}{\partial x} \rightarrow$ differentiating z partially w.r.t 'x'

$\dfrac{\partial z}{\partial y} \rightarrow$ differentiating z partially w.r.t 'y'.

Rules for Partial differentiation:-

Let u and w be functions of x and y.





1) If $z = u + w$ then $\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}$

2) If $z = uw$ then $\frac{\partial z}{\partial x} = u \cdot \frac{\partial w}{\partial x} + w \frac{\partial u}{\partial x}$

3) If $z = \frac{u}{w}$ then $\frac{\partial z}{\partial x} = \frac{w \cdot \frac{\partial u}{\partial x} - u \cdot \frac{\partial w}{\partial x}}{w^2}$

4) If z is a function of 't' and t is a function of x and y

then $\frac{\partial z}{\partial x} = \frac{dz}{dt} \cdot \frac{\partial t}{\partial x}$

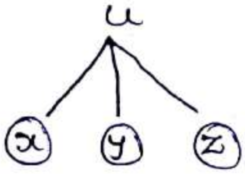
$\frac{\partial z}{\partial y} = \frac{dz}{dt} \cdot \frac{\partial t}{\partial y}$.

Problems :-

① If $u = (x-y)(y-z)(z-x)$ then prove that

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Soln



$u = (x-y)(y-z)(z-x)$.

$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right]$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)(1) + (z-x)(-1)\right]$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

② If $x = r\cos\theta$, $y = r\sin\theta$, find (i) $\frac{\partial r}{\partial x}$ (ii) $\frac{\partial y}{\partial \theta}$

(iii) $\frac{\partial x}{\partial r}$ (iv) $\frac{\partial \theta}{\partial y}$.

Soln:-

$x = r\cos\theta$, $y = r\sin\theta$.

(i) $\frac{\partial x}{\partial r} = \cos\theta$

(ii) $\frac{\partial y}{\partial \theta} = r\cos\theta$.

(iii) $x = r\cos\theta$ $y = r\sin\theta$

$x^2 = r^2\cos^2\theta$ $y^2 = r^2\sin^2\theta$.

$x^2+y^2 = r^2$.

$r = \sqrt{x^2+y^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x$$

$$= \frac{x}{\sqrt{(x^2+y^2)}}$$

(iv) $x = r\cos\theta \qquad y = r\sin\theta$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

$$\frac{y}{x} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2}.$$

③ If $u=x^y$ then show that (i) $\dfrac{\partial^2 u}{\partial y.\partial x} = \dfrac{\partial^2 u}{\partial x.\partial y}$.

(ii) $u_{xxy} = u_{xyx}$.

Soln

$$u=x^y. \quad = e^{\log x^y} \quad = e^{y\log x}$$

(i) $u = e^{y\log x}$.

$$\frac{\partial u}{\partial x} = e^{y\log x} \times y\left(\frac{1}{x}\right)$$

$$= e^{y\log x} . \frac{y}{x}.$$

$$\frac{\partial^2 u}{\partial y.\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(e^{y\log x} \times \frac{y}{x}\right)$$

$$= e^{y\log x}\left(\frac{1}{x}\right) + \frac{y}{x}\left(e^{y\log x} \times \log x\right)$$

$$= \frac{x^y}{x} + \frac{y}{x}\left(x^y \times \log x\right)$$

$$\frac{\partial^2 u}{\partial y . \partial x} = x^{y-1}(1+y\log x) \longrightarrow ①$$

$$u = e^{y\log x}$$

$$\frac{\partial u}{\partial y} = e^{y\log x} \times \log x$$

$$\frac{\partial^2 u}{\partial x . \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(e^{y\log x} \times \log x\right)$$

$$= \log x \times e^{y\log x} \times \frac{y}{x} + e^{y\log x} \times \frac{1}{x}$$

$$= \log x \times x^{y} \times \frac{y}{x} + x^{y} \times \frac{1}{x}$$

$$= x^{y-1}(1+y\log x) \longrightarrow ②$$

From ① and ②

$$\frac{\partial^2 u}{\partial y . \partial x} = \frac{\partial^2 u}{\partial x . \partial y}.$$

(ii) From (i) we have

$$u_{yx} = u_{xy}$$

Diff partially w.r.t 'x'

$$u_{xyx} = u_{xxy}.$$

Hence proved.

Defn (Homogeneous function of degree n)

A function $u(x,y)$ is a homogeneous function of degree n in x and y if

$$u(tx, ty) = t^n u(x,y).$$

Euler Theorem :-

If $u(x,y)$ is a homogenous function of degree 'n' in x and y then

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot u.$$

Note :-

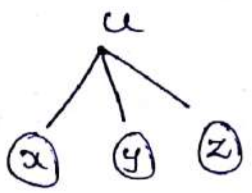
If u is a function of x, y and z then Eulers thm follows that

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = n \cdot u.$$

Problems :-

① If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ then find the value of $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z}$

Soln



Take $u(x,y,z) = \frac{x}{y} + \frac{y}{x} + \frac{z}{x}.$

$$u(tx, ty, tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx}$$

$$= t^{0}\, u(x, y, z).$$

∴ u is a homogenious function of degree '0'

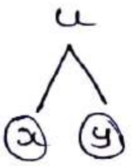
∴ By Eulers thm

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = 0 \times u$$

$$= 0.$$

② If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ then show that $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 0$.

Soln



Take $u(x,y) = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$u(tx, ty) = \sin^{-1}\left(\frac{tx}{ty}\right) + \tan^{-1}\left(\frac{ty}{tx}\right)$$

$$= t^{0}\, u(x,y).$$

∴ u is a homogenious function of degree 0.

$$\Rightarrow \quad x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 0 \times u$$

$$= 0.$$

③ If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then show that

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \tan u.$$

Soln :-

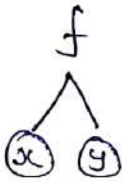
$$u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

$$\Rightarrow \sin u = \frac{x^2+y^2}{x+y}.$$

Take $f(x,y) = \sin u = \frac{x^2+y^2}{x+y}.$

Diff.

Degree



To find degree :-

$$f(tx, ty) = \frac{(tx)^2 + (ty)^2}{tx+ty}$$

$$= t^1 f(x,y).$$

∴ $f(x,y)$ is a homogeneous function of degree 1.

∴ By Eulers thm

$$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = 1 \times f(x,y).$$

$f(x,y) = \sin u.$

$$\frac{\partial f}{\partial x} = \cos u \cdot \frac{\partial u}{\partial x}.$$

$$\frac{\partial f}{\partial y} = \cos u \cdot \frac{\partial u}{\partial y}.$$

$$\therefore \; x \cdot \cos u \cdot \frac{\partial u}{\partial x} + y \cdot \cos u \cdot \frac{\partial u}{\partial y} = \sin u .$$

$$\therefore \; x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \tan u .$$

④ If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \sin 2u .$$

Soln

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$\therefore \; \tan u = \frac{x^3+y^3}{x-y .}$$

Take
$$\begin{tikzcd} f(x,y) = \arrow[r, bend right, "\text{Diff}"'] \arrow[rr, bend right=50, "\text{Degree}"'] & \tan u = & \frac{x^3+y^3}{x-y .} \end{tikzcd}$$

To find degree :-

$$f(x,y) = \frac{x^3+y^3}{x-y .}$$

$$f(tx, ty) = \frac{(tx)^3 + (ty)^3}{tx - ty}$$

$$= t^2 f(x,y) .$$

$\therefore$ $f(x,y)$ is a homogeneous function of degree. '2'.

∴ By Eulers thm

$$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = 2 f(x,y).$$

$$f(x,y) = \tan u.$$

$$\frac{\partial f}{\partial x} = \sec^2 u \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial f}{\partial y} = \sec^2 u \cdot \frac{\partial u}{\partial y}.$$

$$\therefore \quad x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2 \tan u.$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 2 \times \frac{\sin u}{\cos u} \times \frac{1}{\sec^2 u}$$

$$= 2 \times \frac{\sin u}{\cos u} \times \cos^2 u$$

$$= 2 \sin u \cos u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \sin 2u.$$

Hence proved.

⑤ If $u = \frac{x^2+y^2}{\sqrt{x+y}}$ then show that

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{3}{2} u.$$

Soln

$$u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}.$$

$$u(tx, ty) = \frac{(tx)^2 + (ty)^2}{\sqrt{tx+ty}} = t^{3/2} u(x,y)$$

$\therefore$ u is a homogeneous function of degree $\frac{3}{2}$

$\therefore$ By Eulers thm

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{3}{2} \cdot u.$$

Formula:-

If u is a homogeneous function of degree 'n' in x and y then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (n)(n-1)u.$$

Problems

⑥ If $u(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{y}{x}\right)$ then show that $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 2u$.

Soln

$$u(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{y}{x}\right)$$

$$u(tx, ty) = t^2 u(x,y).$$

$\Rightarrow$ u is a homogeneous function of degree 2

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

$$\Rightarrow \quad x^2 \cdot u_{xx} + 2xy \cdot u_{xy} + y^2 u_{yy} = 2(2-1)u$$

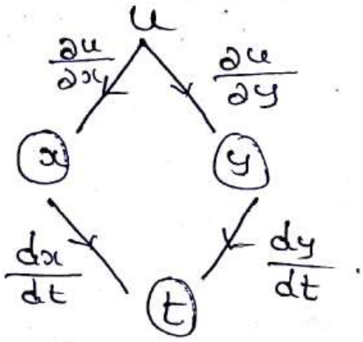
$$= 2u.$$

Hence proved.

Defn (Total Derivative)

If $u = f(x,y)$ where $x = \phi(t)$ and $y = \psi(t)$ then we can express u as a function of 't' alone by substituting the values of x and y in u. Then the ordinary derivative $\frac{du}{dt}$ is called the Total derivative of u.

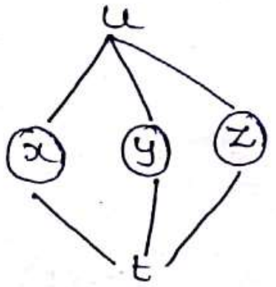
Also $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$.



Problems :-

① If $u = xy + yz + zx$ where $x = \frac{1}{t}$, $y = e^t$, $z = e^{-t}$ then find the value of $\frac{du}{dt}$.

Soln



$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$u = xy + yz + zx$$

$$\frac{\partial u}{\partial x} = y + z \qquad \frac{dx}{dt} = -1/t^2$$

$$\frac{\partial u}{\partial y} = x + z \qquad \frac{dy}{dt} = e^t$$

$$\frac{\partial u}{\partial z} = x + y \qquad \frac{dz}{dt} = -e^{-t}.$$

$$\therefore \frac{du}{dt} = (y+z)(-1/t^2) + (x+z)e^t + (y+x)(-e^{-t})$$

$$= (e^t + e^{-t})(-1/t^2) + (1/t + e^{-t}).e^t$$

$$+ (e^t + 1/t).(-e^{-t})$$

$$= -\frac{(e^t + e^{-t})}{t^2} + \frac{(e^t - e^{-t})}{t}$$

$$= \frac{-2\cosh t}{t^2} + \frac{2\sinh t}{t}.$$

② If $w = f(y-z, z-x, x-y)$ then show that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Soln:-

Given $w = f(y-z, z-x, x-y)$.

Put $y-z = r, \quad z-x = s, \quad x-y = t$

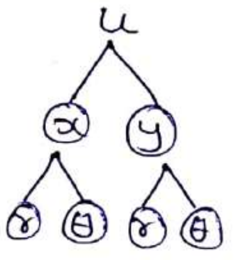
Then $w = f(r, s, t)$.

③ If $u = f(x,y)$ where $x = r\cos\theta$, $y = r\sin\theta$

prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

Soln:-

$x = r\cos\theta$	$y = r\sin\theta$.
$\frac{\partial x}{\partial r} = \cos\theta$.	$\frac{\partial y}{\partial r} = \sin\theta$
$\frac{\partial x}{\partial \theta} = -r\sin\theta$	$\frac{\partial y}{\partial \theta} = r\cos\theta$.



R.H.S $\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y} \cdot \sin\theta .$$

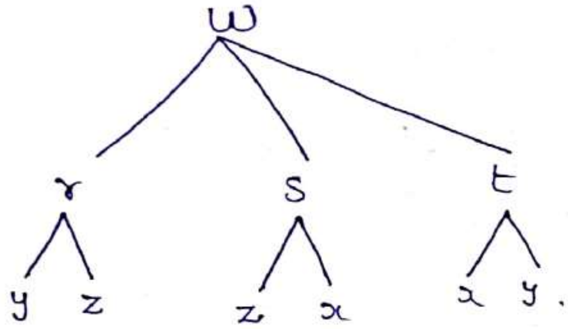
$$\therefore \left(\frac{\partial u}{\partial r}\right)^2 = \left[\cos\theta \cdot \frac{\partial u}{\partial x} + \sin\theta \cdot \frac{\partial u}{\partial y}\right]^2$$

$$= \cos^2\theta \cdot \left(\frac{\partial u}{\partial x}\right)^2 + \sin^2\theta \left(\frac{\partial u}{\partial y}\right)^2 + 2\sin\theta \cdot \cos\theta . \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} . \longrightarrow ①$$

$$\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}\right)^2$$

$$= \left(-r\sin\theta \cdot \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}\right)^2$$

$$= r^2\sin^2\theta \cdot \left(\frac{\partial u}{\partial x}\right)^2 + r^2\cos^2\theta \left(\frac{\partial u}{\partial y}\right)^2 - 2r^2\sin\theta . \cos\theta \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \longrightarrow ②$$



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial w}{\partial s}(-1) + \frac{\partial w}{\partial t}(1)$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial t} - \frac{\partial w}{\partial s} \longrightarrow ①$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$= \frac{\partial w}{\partial r}(1) + \frac{\partial w}{\partial t}(-1)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} - \frac{\partial w}{\partial t} \longrightarrow ②$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial z}$$

$$= \frac{\partial w}{\partial r}(-1) + \frac{\partial w}{\partial s}(1)$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial s} - \frac{\partial w}{\partial r} \longrightarrow ③$$

① + ② + ③

$$\Rightarrow \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} - \frac{\partial w}{\partial s} + \frac{\partial w}{\partial r} - \frac{\partial w}{\partial t} + \frac{\partial w}{\partial s} - \frac{\partial w}{\partial r}$$

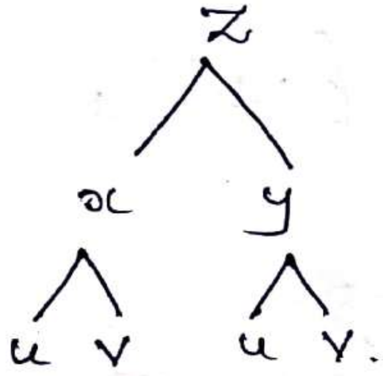
$$= 0.$$

Hence proved.

④ If $z = f(x,y)$, $x = e^{u} \cos v$, $y = e^{u} \sin v$

show that $x . \frac{\partial z}{\partial u} + y . \frac{\partial z}{\partial u} = e^{2u} . \frac{\partial z}{\partial y}$.

Soln



$x = e^{u} \cos v$

$\frac{\partial x}{\partial v} = -e^{u} \sin v.$

$\frac{\partial x}{\partial u} = e^{u} \cos v$

$y = e^{u} \sin v$

$\frac{\partial y}{\partial v} = e^{u} \cos v$

$\frac{\partial y}{\partial u} = e^{u} \sin v.$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

$$= \frac{\partial z}{\partial x}\left(-e^{u} \sin v\right) + \frac{\partial z}{\partial y}\left(e^{u} \cos v\right)$$

$$x \cdot \frac{\partial z}{\partial v} = e^{u} \cos v \left[-\frac{\partial z}{\partial x} \cdot e^{u} \sin v + \frac{\partial z}{\partial y} \cdot e^{u} \cos v\right]$$

$$= -e^{2u} \sin v . \cos v . \frac{\partial z}{\partial x} + e^{2u} \cos^{2} v . \frac{\partial z}{\partial y}.$$

$\longrightarrow$ ①

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

$$= \frac{\partial z}{\partial x} \cdot e^{u} \cos v + \frac{\partial z}{\partial y} \cdot e^{u} \sin v.$$

$$y \cdot \frac{\partial z}{\partial u} = e^{u} \sin v \left[\frac{\partial z}{\partial x} \cdot e^{u} \cos v + \frac{\partial z}{\partial y} \cdot e^{u} \sin v\right]$$

$$= e^{2u} \sin v . \cos v . \frac{\partial z}{\partial x} + e^{2u} \sin^{2} v . \frac{\partial z}{\partial y}$$

$\longrightarrow$ ②

① + ②

$$x \cdot \frac{\partial x}{\partial v} + y \cdot \frac{\partial z}{\partial u} = e^{2u}\left(\sin^{2} v + \cos^{2} v\right) \cdot \frac{\partial z}{\partial y}$$

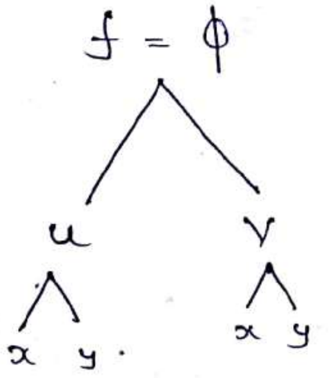
$$= e^{2u} \cdot \frac{\partial z}{\partial y}.$$

L.H.S = R.H.S

Hence proved.

⑤ If $u = x^2 - y^2$, $v = 2xy$, $f(x,y) = \phi(u,v)$ then show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right]$.

Soln:-



$u = x^2 - y^2$	$v = 2xy$.
$\frac{\partial u}{\partial x} = 2x$.	$\frac{\partial v}{\partial x} = 2y$.
$\frac{\partial u}{\partial y} = -2y$	$\frac{\partial v}{\partial y} = 2x$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial \phi}{\partial u} \cdot 2x + \frac{\partial \phi}{\partial v} \cdot 2y \qquad (\because f = \phi)$$

$$\frac{\partial f}{\partial x} = 2x \cdot \frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v} \cdot 2y.$$

$$\therefore \frac{\partial}{\partial x} = 2x \cdot \frac{\partial}{\partial u} + 2y \cdot \frac{\partial}{\partial v}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \left(2x \cdot \frac{\partial}{\partial u} + 2y \cdot \frac{\partial}{\partial v}\right) \cdot \left(2x \cdot \frac{\partial \phi}{\partial u} + 2y \cdot \frac{\partial \phi}{\partial v}\right)$$

$$= 4x^2 \frac{\partial^2 \phi}{\partial u^2} + 4y^2 \frac{\partial^2 \phi}{\partial v^2} + 4xy \frac{\partial^2 \phi}{\partial u \partial v} + 4xy \cdot \frac{\partial^2 \phi}{\partial v \cdot \partial u}$$

$\longrightarrow$ ①

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= -2y \cdot \frac{\partial \phi}{\partial u} + 2x \cdot \frac{\partial \phi}{\partial v}.$$

$$\therefore \frac{\partial}{\partial y} = -2y \cdot \frac{\partial}{\partial u} + 2x \frac{\partial}{\partial v}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \left(-2y \cdot \frac{\partial \phi}{\partial u} + 2x \frac{\partial}{\partial v}\right) \cdot \left(-2y \cdot \frac{\partial \phi}{\partial u} + 2x \cdot \frac{\partial \phi}{\partial v}\right)$$

$$= 4y^2 \frac{\partial^2 \phi}{\partial u^2} + 4x^2 \frac{\partial^2 \phi}{\partial v^2} - 4xy \cdot \frac{\partial^2 \phi}{\partial v . \partial u} - 4xy \frac{\partial^2 \phi}{\partial u \partial}$$

$\hookrightarrow$ ②

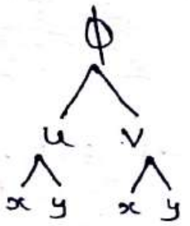
① + ②

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (4x^2 + 4y^2) \frac{\partial^2 \phi}{\partial u^2} + (4x^2 + 4y^2) \frac{\partial^2 \phi}{\partial y^2}$$

$$= 4(x^2+y^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial y^2}\right).$$

⑥ A function $\phi(u,v)$ is written interms of new variables $u = e^x \cos y$, $v = e^x \sin y$ show that $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)$.

Soln:-



$u = e^x \cos y$ | $v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y = u$ | $\frac{\partial v}{\partial x} = e^x \sin y = v.$

$\frac{\partial u}{\partial y} = -e^x \sin y$ | $\frac{\partial v}{\partial y} = e^x \cos y = u.$

$= -v.$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial \phi}{\partial u} \cdot u + \frac{\partial \phi}{\partial v} \cdot v$$

$$\frac{\partial \phi}{\partial x} = u \cdot \frac{\partial \phi}{\partial u} + v \cdot \frac{\partial \phi}{\partial v}.$$

$$\therefore \frac{\partial}{\partial x} = u \cdot \frac{\partial}{\partial u} + v \cdot \frac{\partial}{\partial v}.$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)$$

$$= \left(u \cdot \frac{\partial}{\partial u} + v \cdot \frac{\partial}{\partial v}\right) \cdot \left(u \cdot \frac{\partial \phi}{\partial u} + v \cdot \frac{\partial \phi}{\partial v}\right)$$

$$= u^2 \frac{\partial^2 \phi}{\partial u^2} + v^2 \cdot \frac{\partial^2 \phi}{\partial v^2} + uv \cdot \frac{\partial^2 \phi}{\partial u \cdot \partial v} + uv \cdot \frac{\partial^2 \phi}{\partial v \cdot \partial u} \quad \to ①$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= -v \cdot \frac{\partial \phi}{\partial u} + u \cdot \frac{\partial \phi}{\partial v}.$$

$$\therefore \frac{\partial}{\partial y} = -v \cdot \frac{\partial}{\partial u} + u \cdot \frac{\partial}{\partial v}.$$

$$\therefore \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)$$

$$= \left(-v \cdot \frac{\partial}{\partial u} + u \cdot \frac{\partial}{\partial v}\right) \cdot \left(-v \cdot \frac{\partial \phi}{\partial u} + u \cdot \frac{\partial \phi}{\partial v}\right)$$

$$= v^2 \frac{\partial^2 \phi}{\partial u^2} + u^2 \frac{\partial^2 \phi}{\partial v^2} - uv \cdot \frac{\partial^2 \phi}{\partial u \cdot \partial v} - uv \cdot \frac{\partial^2 \phi}{\partial v \cdot \partial}$$

$$\hookrightarrow ②$$

① + ②

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right).$$

L.H.S = R.H.S

Hence proved.

Jacobians

Defn (Jacobian)

If u and v are functions of two independent variable x and y then Jacobian of u and v with respect to x and y is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ and defined by

$$\frac{\partial(u,v)}{\partial(x,y)} = J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

Similarly if u, v and w are functions of three independent variables x, y and z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobians:-

1) If u and v are function of x and y and x and y are functions of r and θ

then $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$

Problems

① If $x = r\cos\theta$, $y = r\sin\theta$ then find

(i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$

Soln:-

(i) $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$ | $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ | $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ | $\frac{\partial y}{\partial \theta} = r\cos\theta.$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta. \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r.$$

(ii) $$\frac{\partial(r,\theta)}{\partial(x,y)} = ?$$

By property (2)

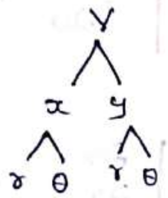
$$\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1.$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}.$$

② If $u = 2xy$, $v = x^2 - y^2$ & $x = r\cos\theta$, $y = r\sin\theta$

evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution.

Soln



$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}.$$

$u = 2xy$	$v = x^2 - y^2$
$\frac{\partial u}{\partial x} = 2y$	$\frac{\partial v}{\partial x} = 2x$
$\frac{\partial u}{\partial y} = 2x$	$\frac{\partial v}{\partial y} = -2y.$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$

$$= -4y^2 - 4x^2$$

$$= -4(x^2+y^2)$$

$$= -4(r^2\cos^2\theta + r^2\sin^2\theta)$$

$$\frac{\partial(u,v)}{\partial(x,y)} = -4r^2.$$ ⟶ ①'

$x = r\cos\theta$ | $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ | $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ | $\frac{\partial y}{\partial \theta} = r\cos\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r. \longrightarrow ②$$

From ① & ②

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4r^2 \times r$$

$$= -4r^3.$$

③ Find the Jacobian of the transformation

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Soln:-

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$x = r\sin\theta\cos\phi$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$y = r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$$

$$\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$z = r\cos\theta.$$

$$\frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial z}{\partial \phi} = 0.$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\left[+r^2\sin^2\theta\cos\phi\right] - r\cos\theta\cos\phi\left[-r\sin\theta\cos\theta\cos\phi\right.$$

$$\left.-r\sin\theta\sin\phi\right[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi\left.\right]$$

$$= +r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi$$

$$+r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi$$

$$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta$$

$$= r^2\sin\theta\left(\sin^2\theta + \cos^2\theta\right)$$

$$= r^2\sin\theta.$$

④ Prove $u = x+y+z$, $v = xy+yz+zx$, $w = x^2+y^2+z^2$ are functionally dependent. Also find the relationship between them.

Soln:-

Functionally dependent condition.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

$u = x+y+z$

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = 1$$

$v = xy + yz + zx$

$$\frac{\partial v}{\partial x} = y+z$$

$$\frac{\partial v}{\partial y} = x+z$$

$$\frac{\partial v}{\partial z} = x+y$$

$w = x^2+y^2+z^2$

$$\frac{\partial w}{\partial x} = 2x$$

$$\frac{\partial w}{\partial y} = 2y$$

$$\frac{\partial w}{\partial z} = 2z.$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & z+x & x+y \\ 2x & 2y & 2z \end{vmatrix}$$

$$= 1\left[2z(z+x) - 2y(x+y)\right]$$

$$-1\left[2z(y+z) - 2x(x+y)\right]$$

$$+1\left[2y(y+z) - 2x(z+x)\right]$$

$$= 2z^2 + 2zx - 2yx - 2y^2$$

$$- 2zy - 2z^2 + 2x^2 + 2xy$$

$$+ 2y^2 + 2yz - 2xz \cancel{+} - 2x^2$$

$$= 0 .$$

$\therefore$ u, v, w are functionally dependent.

W.K.T $(x+y+z)^2 = x^2+y^2+z^2 + 2(xy+yz+zx)$

$\therefore$ $u^2 = w + 2v$ is required

relationship between them.

⑤ Find the jacobian of y_1, y_2 and y_3 w.r.t x_1, x_2, x_3 if $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_2 x_1}{x_3}$.

Soln:-

$$y_1 = \frac{x_2 x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2}$$

$$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$y_2 = \frac{x_3 x_1}{x_2}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$$

$$\frac{\partial y_2}{\partial x_2} = -\frac{x_3 x_1}{x_2^2}$$

$$\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$y_3 = \frac{x_2 x_1}{x_3}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$$

$$\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$$

$$\frac{\partial y_3}{\partial x_3} = -\frac{x_2 x_1}{x_3^2}.$$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_2 x_1}{x_3^2} \end{vmatrix}$$

$$= -\frac{x_2 x_3}{x_1^2}\left[\frac{x_1^2 x_2 x_3}{x_2^2 x_3^2} - \frac{x_1^2}{x_2 x_3}\right] - \frac{x_3}{x_1}\left[-\frac{x_1 x_2 x_3}{x_2 x_3^2} - \frac{x_1 x_2}{x_2 x_3}\right]$$

$$+ \frac{x_2}{x_1}\left[\frac{x_1 x_3}{x_2 x_3} + \frac{x_1 x_2 x_3}{x_3 x_2^2}\right]$$

$$= \frac{-x_2 x_3}{x_1^2}\left[\frac{x_1^2}{x_2 x_3} - \frac{x_1^2}{x_2 x_3}\right] - \frac{x_3}{x_1}\left[-\frac{x_1}{x_3} - \frac{x_1}{x_3}\right]$$

$$+ \frac{x_2}{x_1}\left[\frac{x_1}{x_2} + \frac{x_1}{x_2}\right]$$

$$= 0 - \frac{x_3}{x_1}\left(-\frac{2x_1}{x_3}\right) + \frac{x_2}{x_1}\left(2\frac{x_1}{x_2}\right)$$

$$= 2 + 2$$

$$= 4.$$

Taylor's Series for Function of Two Variables:

Formula:-

If $f(x,y)$ is a function of two variables x and y then the Taylor Series expansion of $f(x,y)$ about the point (a,b) is

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[A f_x(a,b) + B f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[A^2 f_{xx}(a,b) + 2AB f_{xy}(a,b) + B^2 f_{yy}(a,b)\right]$$

$$+ \frac{1}{3!}\left[A^3 f_{xxx}(a,b) + 3A^2B f_{xxy}(a,b) + 3AB^2 f_{xyy}(a,b) + B^3 f_{yyy}(a,b)\right]$$

$+ \cdots$ where $A = (x-a)$

& $B = (y-b)$.

Problems :-

① Obtain terms upto third degree in the taylor series expansion of $e^x \sin y$ about the point $(1, \frac{\pi}{2})$.

Soln :-

Function	Value at $(1, \frac{\pi}{2})$
$f(x,y) = e^x \sin y$	$f(a,b) = e$
$f_x = e^x \sin y$	$f_x(a,b) = e$
$f_y = e^x \cos y$	$f_y(a,b) = 0$
$f_{xx} = e^x \sin y$	$f_{xx}(a,b) = e$
$f_{xy} = e^x \cos y$	$f_{xy}(a,b) = 0$
$f_{yy} = -e^x \sin y$	$f_{yy}(a,b) = -e$
$f_{xxx} = e^x \sin y$	$f_{xxx}(a,b) = e$
$f_{xxy} = e^x \cos y$	$f_{xxy}(a,b) = 0$
$f_{xyy} = -e^x \sin y$	$f_{xyy}(a,b) = -e$
$f_{yyy} = -e^x \cos y$	$f_{yyy}(a,b) = 0$.

∴ Taylor Series is $A = (x-1)$, $B = (y - \pi/2)$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[A f_x(a,b) + B f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[A^2 f_{xx}(a,b) + 2AB f_{xy}(a,b) + B^2 f_{yy}(a,b)\right]$$

$$+ \frac{1}{3!}\left[A^3 f_{xxx}(a,b) + 3A^2B f_{xxy}(a,b) + 3AB^2 f_{xyy}(a,b) + B^3 f_{yyy}(a,b)\right]$$

$$+ \cdots$$

$$\therefore \; e^{x}\sin y = e + \frac{1}{1!}\left[e(x-1) + 0(y-\pi/2)\right]$$

$$+\frac{1}{2!}\left[e(x-1)^2 + 2(0)(x-1)(y-\pi/2) + (-e)(y-\pi/2)^2\right]$$

$$+\frac{1}{3!}\left[e(x-1)^3 + 3(0)(x-1)^2(y-\pi/2) + 3(-e)(x-1)(y-\pi/2)^2 + 0(y-\pi/2)^3\right] + \dots$$

$$e^{x}\sin y = e + e(x-1) + \frac{1}{2!}\left[e(x-1)^2 - e(y-\pi/2)^2\right]$$

$$+\frac{1}{3!}\left[e(x-1)^3 - 3e(x-1)(y-\pi/2)^2\right] + \dots$$

② Expand $e^{x}\cos y$ about $(0, \frac{\pi}{2})$ upto 3^{rd} degree terms using Taylor Series.

Soln.

Function	Value at $(0, \frac{\pi}{2})$
$f(x,y) = e^{x}\cos y$	$f(a,b) = 0$
$f_x = e^{x}\cos y$	$f_x(a,b) = 0$
$f_y = -e^{x}\sin y$	$f_y(a,b) = -1$
$f_{xx} = e^{x}\cos y$	$f_{xx}(a,b) = 0$
$f_{xy} = -e^{x}\sin y$	$f_{xy}(a,b) = -1$
$f_{yy} = -e^{x}\cos y$	$f_{yy}(a,b) = 0$
$f_{xxx} = e^{x}\cos y$	$f_{xxx}(a,b) = 0$
$f_{xxy} = -e^{x}\sin y$	$f_{xxy}(a,b) = -1$
$f_{xyy} = -e^{x}\cos y$	$f_{xyy}(a,b) = 0$
$f_{yyy} = e^{x}\sin y$	$f_{yyy}(a,b) = 1.$

$$A = (x-0)$$

$$B = (y - \pi/2).$$

∴ Taylor Series expansion is

$$e^{x}\cos y = \frac{1}{1!}\left[0 - 1(B)\right] + \frac{1}{2!}\left[0 + 2(-1)AB + 0\right]$$

$$+ \frac{1}{3!}\left[0 + 3(-1)A^{2}B + 3(0)AB^{2} + 1B^{3}\right] + \cdots$$

$$= \frac{1}{1!}\left[-1\left(y-\pi/2\right)\right] + \frac{1}{2!}\left[-2(x)\left(y-\pi/2\right)\right]$$

$$+ \frac{1}{3!}\left[-3(x)^{2}(y-\pi/2)^{2} + (y-\pi/2)^{3}\right] + \cdots$$

③ Expand $x^2y + 3y - 2$ inpowers of $(x-1)$ & $(y+2)$ upto third degree terms using Taylor Series.

Soln

Function	Value at $(1,-2)$
$f(x,y) = x^2y + 3y - 2$	$f(1,-2) = -10$
$f_x = 2xy$	$f_x(1,-2) = -4$
$f_y = x^2+3$	$f_y(1,-2) = 4$
$f_{xx} = 2y$	$f_{xx}(1,-2) = -4$
$f_{xy} = 2x$	$f_{xy}(1,-2) = 2$
$f_{yy} = 0$	$f_{yy}(1,-2) = 0$
$f_{xxx} = 2$	$f_{xxx}(1,-2) = 0$
$f_{xxy} = 2$	$f_{xxy}(1,-2) = 2$
$f_{xyy} = 0$	$f_{xyy}(1,-2) = 0$
$f_{yyy} = 0$	$f_{yyy}(1,-2) = 0$

∴ Taylor Series expansion is

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[f_x A + \cancel{2A}B f_y\right]$$
$$+ \frac{1}{2!}\left[f_{xx} A^2 + 2AB f_{xy} + B^2 f_{yy}\right]$$
$$+ \frac{1}{3!}\left[A^3 f_{xxx} + 3A^2B f_{xxy} + 3AB^2 f_{xyy} + B^3 f_{yyy}\right]$$
$$+ \ldots$$

Here $A = (x-1)$

$B = (y+2)$

$$\therefore\ x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2$$
$$+ 2(x-1)(y+2) + 2(x-1)^2(y+2) + \ldots$$

④ Expand $e^x \log(1+y)$ in powers of x and y upto third degree terms.

Soln.

$A = (x-0) = x$

$B = (y-0) = y.$

Point → $(0,0)$.

Taylor Series expansion

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[A f_x + B f_y\right] + \frac{1}{2!}\left[A^2 f_{xx} + 2AB f_{xy} + B^2 f_{yy}\right]$$
$$+ \frac{1}{3!}\left[A^3 f_{xxx} + 3A^2B f_{xxy} + 3AB^2 f_{xyy} + B^3 f_{yyy}\right]$$
$$+ \ldots$$

Function	Value at (0,0).
$f(x,y) = e^{x}\log(1+y)$	$f_{x}(a,b) = 0$
$f_x = e^{x}\log(1+y)$	$f_x(a,b) = 0$
$f_y = e^{x}(1+y)^{-1}$	$f_y(a,b) = 1$
$f_{xx} = e^{x}\log(1+y)$	$f_{xx}(a,b) = 0$
$f_{xy} = e^{x}(1+y)^{-1}$	$f_{xy}(a,b) = 1.$
$f_{yy} = -e^{x}(1+y)^{-2}$	$f_{yy}(a,b) = -1.$
$f_{xxx} = e^{x}\log(1+y)$	$f_{xxx}(a,b) = 0$
$f_{xxy} = e^{x}(1+y)^{-1}$	$f_{xxy}(a,b) = 1.$
$f_{xyy} = -e^{x}(1+y)^{-2}$	$f_{xyy}(a,b) = -1.$
$f_{yyy} = 2e^{x}(1+x)^{-3}$	$f_{yyy}(a,b) = 2.$

$\therefore$ Taylor series expansion is

$$e^{x}\log(1+y) = y + \frac{1}{2!}\left[2xy - y^{2}\right] + \frac{1}{3!}\left[3x^{2}y - 3xy^{2} + 2y^{3}\right]$$

$$+ \cdots$$

⑤ Expand $\sin(xy)$ in powers of $(x-1)$ and $(y-\frac{\pi}{2})$ upto second degree terms using Taylor Series.

Soln:-

Function	Value at $(1, \pi/2)$
$f(x,y) = \sin(xy)$	$f(a,b) = 1$
$f_x = y\cos xy$	$f_x(a,b) = 0$
$f_y = x\cos xy$	$f_y(a,b) = 0$
$f_{xx} = -y\sin(xy) \times y$ $= -y^2 \sin xy$	$f_{xx}(a,b) = -\pi^2/4$.
$f_{xy} = xy[-\sin xy] + \cos xy$	$f_{xy}(a,b) = -\pi/2$
$f_{yy} = x^2[-\sin xy]$	$f_{yy}(a,b) = -1$.
$A = (x-1)$	$B = (y-\pi/2)$

Taylor Series expansion is

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[A f_x + B f_y\right] + \frac{1}{2!}\left[A^2 f_{xx} + 2AB f_{xy} + B^2 f_{yy}\right] + \ldots$$

$$\Rightarrow \sin(xy) = 1 + \frac{1}{1!}\left[A(0) + B(0)\right] + \frac{1}{2!}\left[-A^2\pi^2/4 - 2AB(-\pi/2) + B^2(-1)\right] + \ldots$$

$$\sin xy = 1 + \frac{1}{2}\left[-(x-1)^2\pi^2/4 + (x-1)(y-\pi/2)\pi \cancel{+} - (y-\pi/2)^2\right]$$

Maxima and Minima of Functions of Two Variables :-

Extremum Value → Either maximum value or minimum value.

Saddle point → Neither maximum nor minimum point

Critical point → Stationary point.

Method of finding Maxima and Minima :-

① Take the given function as $f(x,y)$

② Find f_x, f_y, $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

③ Find Stationary points from $\boxed{\begin{matrix} f_x = 0 \\ \& \\ f_y = 0 \end{matrix}}$

④ Find $AC - B^2$ for each of the stationary points.

Values	Conclusion
$AC - B^2 > 0$ and $(A < 0$ or $B < 0)$	Maximum
$AC - B^2 > 0$ and $(A > 0$ or $B > 0)$	Minimum
$AC - B^2 < 0$	Saddle point
$AC - B^2 = 0$	Inconclusive.

Problems :-

① Find extremum values for the function

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Soln:-

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3$

$f_y = 3y^2 - 12$

$A = f_{xx} = 6x$

$B = f_{xy} = 0$

$C = f_{yy} = 6y$.

$f_x = 0$	$f_y = 0$
$3x^2 - 3 = 0$	$3y^2 - 12 = 0$
$3(x^2 - 1) = 0$.	$3(y^2 - 4) = 0$
$x^2 - 1 = 0$	$y^2 - 4 = 0$
$x = \pm 1$	$y = \pm 2$.

∴ Stationary points are $(1, 2)$ $(-1, 2)$ $(-1, -2)$ $(1, -2)$.

	At (1, 2)	At (1, −2)	At (−1, 2)	At (−1, −2).
$A = 6x$	$6 > 0$	6	-6	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
conclusion	Minimum point	Saddle point	Saddle point	Maximum point.

$\therefore$ $f(x,y)$ takes the maximum value at $(-1, -2)$

" " " minimum value at $(1, 2)$.

$\therefore$ Maximum value of $f(x,y) = \Big[f(x,y)\Big]_{\text{Put } x=-1,\ y=-2}$

$$= \left(x^3+y^3-3x-12y+20\right)_{x=-1,\ y=-2}$$

$$= -1-8+3+24+20$$

$$= 38.$$

Minimum value of $f(x,y) = \Big[f(x,y)\Big]_{\text{Put } (x=1,\ y=2)}$

$$= \left(x^3+y^3-3x-12y+20\right)_{\text{Put } x=1,\ y=2}$$

$$= 1+8-3-24+20$$

$$= 2.$$

② Examine $x^3y^2(12-x-y)$ for extremum values.

Soln

$$f(x,y) = x^3y^2(12-x-y)$$

$$f(x,y) = 12x^3y^2 - x^4y^2 - x^3y^3.$$

$$f_x = 36x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 24x^3y - 2x^4y - 3x^3y^2$$

$$A = f_{xx} = 72xy^2 - 12x^2y^2 - 6xy^3$$

$$B = f_{xy} = 72x^2y - 8x^3y - 9x^2y^2$$

$$C = f_{yy} = 24x^3 - 2x^4 - 6x^3y$$

$f_x = 0$

$$36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(36-4x-3y) = 0.$$

$x^2y^2 = 0$ (or) $36-4x-3y=0$

$\Rightarrow$ $x=0$ (or) $y=0$, $4x+3y=36 \rightarrow$ ①

$f_y = 0$

$$24x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^3y(24-2x-3y) = 0$$

$x=0$, $y=0$, $3y+2x=24. \rightarrow$ ②

① − ②

$\Rightarrow$ $2x = 12$

$$\boxed{x = 6}$$

Sub $x = 6$ in ②

$3y + 12 = 24.$

$3y = 12$

$$\boxed{y = 4}$$

∴ Stationary points are $(0,0)$, $(6,4)$.

	At (0,0)	At (6,4)
$A = 72xy^2 - 12x^2y^2 - 6xy^3$	0	$-2304 < 0$
$B = 72x^2y - 8x^3y - 9x^2y^2$	0	$-5184 < 0$
$C = 24 - 2x^4 - 6x^3y$	24	$-7752 < 0$
$AC - B^2$	0	$AC - B^2 > 0$
Conclusion	Inconclusive	Maximum point

$\therefore$ $f(x,y)$ takes the maximum value at $(6,4)$.

Maximum value of $f(x,y) = \left[x^3y^2(12-x-y)\right]_{(6,4)}$

$$= 6^3 4^2 (12-6-4)$$

$$= 216 \times 16 \times 2$$

$$= 6912.$$

③ A flat circular plate is heated so that the temperature at any point (x,y) is $u(x,y) = x^2 + 2y^2 - x$. Find the coldest point on the plate.

Soln:-

Take temperature = function

cold = temp. minimum = function minimum.

Let $f(x,y) = u(x,y) = x^2 + 2y^2 - x.$

$f_x = 2x - 1.$

$f_y = 4y.$

$A = f_{xx} = 2$

$B = f_{xy} = 0$

$C = f_{yy} = 4.$

$f_x = 0$	$f_y = 0$
$\Rightarrow 2x - 1 = 0$	$4y = 0$
$\boxed{x = 1/2}$	$\boxed{y = 0}$

$\therefore$ Stationary point $(1/2, 0)$.

At $(1/2, 0)$ $\therefore AC - B^2 = 8 > 0$

$A = 2 > 0$.

$\therefore$ $f(x,y)$ takes the minimum value at $(\frac{1}{2}, 0)$.

$\therefore$ $(\frac{1}{2}, 0)$ is the coldest point.

The temperature at $(\frac{1}{2}, 0)$ $= 1/4 + 2(0) - 1/2$

$= -1/4$.

④ Discuss the maxima and minima of the function $f(x,y) = x^3 + y^3 - 3axy$.

Soln :-

$f(x,y) = x^3 + y^3 - 3axy.$

$f_x = 3x^2 - 3ay.$

$f_y = 3y^2 - 3ax.$

$A = f_{xx} = 6x$

$B = f_{xy} = -3a$

$C = f_{yy} = 6y$

$f_x = 0$	$f_y = 0$
$3x^2 - 3ay = 0$	$3y^2 - 3ax = 0$
$x^2 = ay$	$y^2 = ax$
$\boxed{\frac{x^2}{a} = y} \longrightarrow$ ①	$\boxed{x = y^2/a}$ $\hookrightarrow$ ②

Sub $y = x^2/a$ in ②

$$x = \frac{x^4}{a^3}$$

$$1 = x^3/a^3.$$

$$x^3 = a^3$$

$$\Rightarrow \boxed{x = a}$$

Sub $x = a$ in ①

$$y = a^2/a$$

$$\boxed{y = a}$$

∴ Stationary point is (a, a).

	At (a, a)
$A = 6x$	$6a$
$B = -3a$	$-3a$
$C = 6y$	$6a$
$AC - B^2$	$27a^2 > 0$

Clearly $27a^2 > 0$.

But $A = 6a > 0$ if $a > 0$

Also $A = 6a < 0$ if $a < 0$.

$\therefore$ If $a > 0$, (a, a) is minimum point

If $a < 0$, (a, a) is maximum point.

Maximum (or) minimum value of

$$f(x,y) = a^3 + a^3 - 3a(a)(a)$$

$$= -a^3.$$

⑤ Find the maxima and minima of $xy(a-x-y)$

Soln:-

$$f(x,y) = xy(a-x-y).$$

$$= axy - x^2y - xy^2$$

$$f_x = ay - 2xy - y^2$$

$$f_y = ax - x^2 - 2xy.$$

$A = f_{xx} = -2y$

$B = f_{xy} = a - 2x - 2y$

$C = f_{yy} = -2x.$

$f_x = 0$	$f_y = 0$
$ay - 2xy - y^2 = 0.$	$ax - x^2 - 2yx = 0$
$y(a - 2x - y) = 0.$	$x(a - x - 2y) = 0$
$y = 0$ (or) $a - 2x - y = 0$ $\hookrightarrow$ ①	$x = 0$ (or) $a - x - 2y = 0$ $\hookrightarrow$ ②

From ① & ②

$$2x + y = a \longrightarrow ③$$

$$x + 2y = a \longrightarrow ④$$

③ $-$ 2(④) $\Rightarrow$

$$\begin{array}{r} 2x + y = a \\ (-)\,2x \overset{(-)}{+} 4y = \overset{(-)}{2a} \\ \hline -3y = -a \end{array}$$

$$\boxed{y = a/3}$$

From ①

$$2x + a/3 = a.$$

$$2x = a - a/3$$

$$x = \frac{2a}{3 \times 2}$$

$$\boxed{x = a/3}$$

$\therefore$ Stationary points are $(0,0)$ & $(a/3, a/3)$.

	At $(0,0)$	At $\left(\frac{a}{3}, \frac{a}{3}\right)$
$A = -2y$	0	$-2a/3$
$B = a-2x-2y$	a	$-a/3$
$C = -2x$	0	$-2a/3$
$AC-B^2$	$-a^2 < 0$	$a^2/3 > 0$.
Conclusion	Saddle point	

If $a<0$, $\left(\frac{a}{3}, \frac{a}{3}\right) \rightarrow$ minimum point

If $a>0$, $\left(\frac{a}{3}, \frac{a}{3}\right) \rightarrow$ maximum point.

$\therefore$ Minimum (or) Maximum value of $f(x,y)$

$$= a\left(\frac{a}{3}\right)\left(\frac{a}{3}\right) - \left(\frac{a}{3}\right)^2\left(\frac{a}{3}\right) - \left(\frac{a}{3}\right)^2\left(\frac{a}{3}\right)$$

$$= a^3/9 - a^3/27 - a^3/27$$

$$= \frac{3a^3 - 2a^3}{27}$$

$$= a^3/27.$$

Lagranges Method of undetermined Multiplier :-

When appey this method ?.

To find the maxima (or) minima of a function $f(x,y,z)$ of three variables subject to the constraint eqn $g(x,y,z)=0$.

Method :-

1) Take function $= f(x,y,z)$

Constraint eqn $= g(x,y,z)$

2) Take $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$.

Here $\lambda \rightarrow$ Lagrange multiplier.

3) Find F_x, F_y and F_z.

4) Find the point (x,y,z) from $\boxed{\begin{array}{c} F_x = 0 \\ F_y = 0 \\ \& \\ F_z = 0 \end{array}}$

5) Given $f(x,y,z)$ is maximum (or) minimum at (x,y,z).

Problems :-

① Find the minimum value of $x^2+y^2+z^2$ subject to the condition. $x+y+z = 3a$.

Soln:-

$$f(x,y,z) = x^2+y^2+z^2$$

Condition :- $x+y+z = 3a$

$$x+y+z-3a = 0$$

$$g(x,y,z) = x+y+z-3a.$$

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$= x^2+y^2+z^2+ \lambda(x+y+z-3a)$$

$$= x^2+y^2+z^2+\lambda x+\lambda y+\lambda z-3a\lambda$$

$$F_x = 2x+\lambda$$

$$F_y = 2y+\lambda$$

$$F_z = 2z+\lambda.$$

$F_x = 0$ $\Rightarrow 2x+\lambda = 0$ $x = -\lambda/2 \rightarrow ①$	$F_y = 0$ $\Rightarrow 2y+\lambda = 0$ $y = -\lambda/2 \rightarrow ②$	$F_z = 0$ $\Rightarrow 2z+\lambda = 0$ $z = -\lambda/2. \rightarrow ③$

From ①, ② & ③

$$x = y = z.$$

Given $x+y+z = 3a$

$$\Rightarrow x+x+x = 3a$$

$$\Rightarrow \boxed{x = a}$$

$$\therefore \boxed{y = a} \quad \& \quad \boxed{z = a}$$

$\therefore$ Minimum point $\rightarrow (x, y, z) \rightarrow (a, a, a)$

Minimum value of $f(x,y,z) = \left[x^2+y^2+z^2\right]$

Put $x=a, y=a, z=a$

$= a^2+a^2+a^2$

$= 3a^2.$

② Find the maximum value of $x^m y^n z^p$ subject to the condition $x+y+z=a$.

Soln :-

$f(x,y,z) = x^m y^n z^p$

Condition :- $x+y+z=a$.

$x+y+z-a=0$.

$g(x,y,z) = x+y+z-a$

$\therefore F(x,y,z) = x^m y^n z^p + \lambda(x+y+z-a)$

$= x^m y^n z^p + x\lambda + y\lambda + z\lambda - a\lambda.$

$F_x = mx^{m-1} y^n z^p + \lambda.$

$F_y = nx^m y^{n-1} z^p + \lambda$

$F_z = px^m y^n z^{p-1} + \lambda.$

$F_x = 0$	$F_y = 0$	$F_z = 0$
$\Rightarrow mx^{m-1}y^n z^p + \lambda = 0$	$nx^m y^{n-1} z^p + \lambda = 0$	$Px^m y^n z^{P-1} + \lambda = 0$
$mx^{m-1}y^n z^p = -\lambda$	$nx^m y^{n-1} z^p = -\lambda$	$Px^m y^n z^{P-1} = -\lambda$
$\hookrightarrow$ ①	$\hookrightarrow$ ②	$\hookrightarrow$ ③

From ①, ② & ③

$$m x^{m-1} y^{n} z^{p} = n x^{m} y^{n-1} z^{p} = p x^{m} y^{n} z^{p-1}$$

ie) $$\frac{m x^{m} y^{n} z^{p}}{x} = \frac{n x^{m} y^{n} z^{p}}{y} = \frac{p x^{m} y^{n} z^{p}}{z}.$$

$$\Rightarrow \quad \frac{m}{x} = \frac{n}{y} = \frac{p}{z}.$$

∴ ~~$x = my$~~

$$\frac{m}{x} = p/z. \qquad \& \qquad \frac{n}{y} = p/z.$$

$$\Rightarrow \quad x = \frac{mz}{p} \longrightarrow ④ \qquad \& \qquad y = \frac{nz}{p}. \longrightarrow ⑤$$

Given $x + y + z = a.$

$$\frac{mz}{p} + \frac{nz}{p} + z = a.$$

$$z\left(\frac{m}{p} + \frac{n}{p} + 1\right) = a.$$

$$z \times \left(\frac{m+n+p}{p}\right) = a.$$

$$\boxed{z = \frac{ap}{m+n+p}}$$

∴ From ④

$$x = \frac{m}{p}\left(\frac{ap}{m+n+p}\right)$$

$$\boxed{x = \frac{am}{m+n+p}}$$

From ⑤ $$y = \frac{n}{p} \times \frac{ap}{m+n+p} = \frac{an}{m+n+p}.$$

$\therefore$ Maximum point $\rightarrow (x,y,z) \rightarrow \left(\frac{am}{m+n+p}, \frac{an}{m+n+p}, \frac{ap}{m+n+p}\right)$

$\therefore$ Maximum Value of $f(x,y,z)$

$$= \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$= \frac{a^{m+n+p} \cdot m^m n^n p^p}{(m+n+p)^{m+n+p}}.$$

③ Find the three positive numbers such that their Sum is a constant 'a' and their product is maximum.

Soln:-

Required numbers be x, y, z.

xyz is maximum subject to the condition $x+y+z=a$.

$$f(x,y,z) = xyz$$

$$g(x,y,z) = x+y+z-a.$$

$$F(x,y,z) = f+\lambda g$$

$$= xyz+\lambda x+\lambda y+\lambda z-a\lambda.$$

$F_x = yz + \lambda$

$F_y = xz + \lambda$

$F_z = xy + \lambda$.

$F_x = 0$	$F_y = 0$	$F_z = 0$
$\Rightarrow yz + \lambda = 0$	$\Rightarrow xz + \lambda = 0$	$\Rightarrow xy + \lambda = 0$
$yz = -\lambda$	$xz = -\lambda$	$xy = -\lambda$.
↳ ①	↳ ②	↳ ③

From ①, ② & ③

$yz = xz = xy$.

$\Rightarrow yz = xz$ & $xz = xy$

$\Rightarrow y = x$ & $z = y$.

$\Rightarrow x = y = z$.

condition $x + y + z = a$.

$\Rightarrow 3x = a$

$\Rightarrow \boxed{x = a/3}$

$\therefore \boxed{y = a/3}$ & $\boxed{z = a/3}$

∴ Maximum point → $(x, y, z) \rightarrow (a/3, a/3, a/3)$

Maximum Value of $f(x, y, z) = a/3 \times a/3 \times a/3$

$= a^3/27$.

④ A rectangular box open at the top is to have a volume 32cc. Find the dimension of the box that requires least material for its construction.

Soln:-

Length = x cm

Breadth = y cm

Height = z cm.

Condition $xyz = 32$.

Surface Area = $2(xy + yz + zx)$

Top open Surface area $= xy + 2yz + 2zx$.

$$f(x,y,z) = xy + 2yz + 2zx.$$

$$g(x,y,z) = xyz - 32.$$

$$F(x,y,z) = f + \lambda g$$

$$= (xy + 2yz + 2zx) + \lambda(xyz - 32)$$

$$F(x,y,z) = xy + 2yz + 2zx + \lambda xyz - 32\lambda.$$

$$F_x = y + 2z + \lambda yz$$

$$F_y = x + 2z + \lambda xz$$

$$F_z = 2y + 2x + \lambda xy.$$

$F_x = 0$

$y + 2z + \lambda yz = 0$

$y + 2z = -\lambda yz.$

$\frac{y+2z}{yz} = -\lambda$

$\frac{1}{z} + \frac{2}{y} = -\lambda \longrightarrow$ ①

$F_y = 0$

$x + 2z + \lambda xz = 0$

$\frac{x+2z}{xz} = -\lambda.$

$\frac{1}{z} + \frac{2}{x} = -\lambda \longrightarrow$ ②

$F_z = 0$

$2y + 2x + \lambda xy = 0$

$2y + 2x = -\lambda xy$

$\frac{2}{x} + \frac{2}{y} = -\lambda \longrightarrow$ ③

From ① & ②

$$\frac{2}{y} = 2/x.$$

$$\Rightarrow \boxed{x = y}$$

From ② & ③

$$1/z = 2/y.$$

$$\boxed{y = 2z}$$

∴ Given condition

$$xyz = 32.$$

$$y \times y \times y/2 = 32.$$

$$y^3 = 64.$$

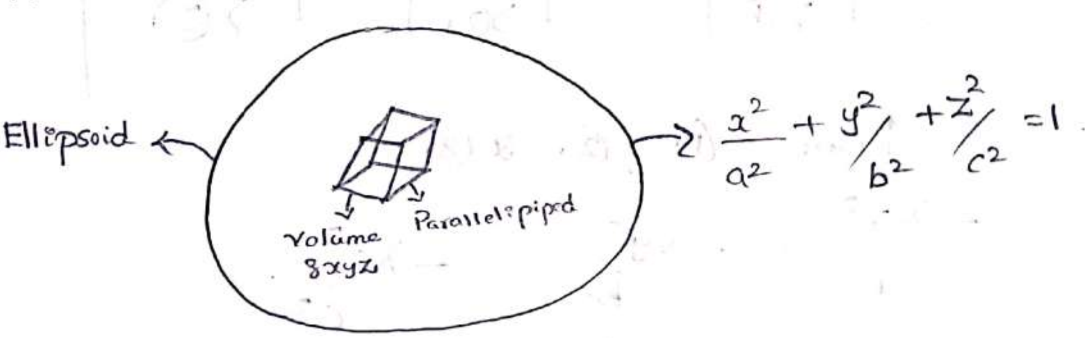
$$\boxed{y = 4}$$

$$\therefore \boxed{x = 4} \quad \boxed{z = 2}$$

∴ Required dimensions are 4cm, 4cm, 2cm.

⑤ Find the maximum volume of the largest rectangular parallelepiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2} + y^2/b^2 + z^2/c^2 = 1$.

Soln :-



$f(x,y,z) = 8xyz$.

Condition : $\frac{x^2}{a^2} + y^2/b^2 + z^2/c^2 = 1$.

$\therefore \quad g(x,y,z) = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1$.

$$F(x,y,z) = f + \lambda g$$

$$= 8xyz + \lambda\left(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1\right)$$

$$= 8xyz + \lambda x^2/a^2 + \lambda y^2/b^2 + \lambda z^2/c^2 - \lambda .$$

$$F_x = 8yz + \frac{2\lambda x}{a^2}$$

$$F_y = 8xz + \frac{2\lambda y}{b^2}$$

$$F_z = 8xy + \frac{2\lambda z}{c^2}.$$

$F_x = 0$	$F_y = 0$	$F_z = 0$
$8yz = -\frac{2\lambda x}{a^2}.$	$8xz = -\frac{2\lambda y}{b^2}$	$8xy = -\frac{2\lambda z}{c^2}$
$4yz = -\frac{2\lambda x}{a^2}$	$4xz = -\frac{2\lambda y}{b^2}$	$4xy = -\lambda z/c^2$
$\therefore 4xyz = \frac{-\lambda x^2}{a^2}$	$4xyz = -\frac{\lambda y^2}{b^2}$	$4xyz = -\lambda z^2/c^2$
$\hookrightarrow$ ①	$\hookrightarrow$ ②	$\hookrightarrow$ ③

From ① , ② & ③

$$\frac{-\lambda x^2}{a^2} = -\lambda y^2/b^2 = -\lambda z^2/c^2.$$

$$\therefore \quad \frac{x^2}{a^2} = y^2/b^2 = z^2/c^2$$

Given $\frac{x^2}{a^2} + y^2/b^2 + z^2/c^2 = 1$.

$$\therefore \quad x^2/a^2 + x^2/a^2 + x^2/a^2 = 1.$$

$$\frac{3x^2}{a^2} = 1.$$

$$x^2 = a^2/3.$$

$$\boxed{x = a/\sqrt{3}}$$

$$\therefore \quad \boxed{y = b/\sqrt{3}} \;, \quad \boxed{z = c/\sqrt{3}}$$

$\therefore$ Minimum point $\rightarrow$ $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$.

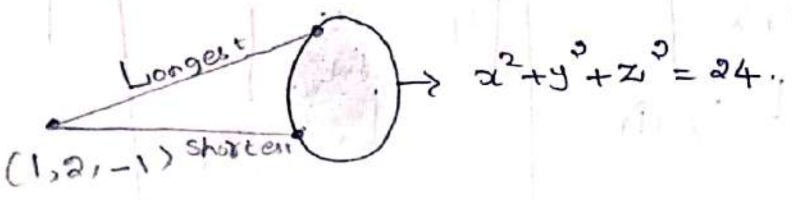
Maximum Volume $= \boxed{8xyz}$ Put $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, $z = c/\sqrt{\ }$

$$= 8 \times \frac{a}{\sqrt{3}} \times \frac{b}{\sqrt{3}} \times \frac{c}{\sqrt{3}}$$

$$= \frac{8abc}{3\sqrt{3}}$$

8. Find the shortest and longest distances from the point $(1,2,-1)$ to the sphere $x^2+y^2+z^2 = 24$ using Lagrange method of constrained maxima and minima.

Soln:-



Let (x,y,z) be any point on the sphere $x^2+y^2+z^2 = 24$.

Aim:- $d = \sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$ is maximum (or) minimum

Equivalently $d^2 = (x-1)^2+(y-2)^2+(z+1)^2$ is maximum (or) minimum.

$$f(x,y,z) = (x-1)^2+(y-2)^2+(z+1)^2.$$

$$g(x,y,z) = x^2+y^2+z^2-24.$$

$$\therefore F(x,y,z) = f+\lambda g$$

$$= (x-1)^2+(y-2)^2+(z+1)^2$$
$$+\lambda x^2+\lambda y^2+\lambda z^2-24\lambda.$$

$$F_x = 2(x-1)+2\lambda x$$

$$F_y = 2(y-2)+2\lambda y$$

$$F_z = 2(z+1)+2\lambda z.$$

$$F_x = 0$$
$$2(x-1) = -2\lambda x.$$
$$x-1 = -\lambda x.$$
$$x+\lambda x = 1.$$
$$\boxed{x = \frac{1}{1+\lambda}} \rightarrow ①$$

$$F_y = 0$$
$$2(y-2) = -2\lambda y$$
$$y-2 = -\lambda y.$$
$$y+\lambda y = 2.$$
$$\boxed{y = \frac{2}{1+\lambda}} \rightarrow ②$$

$$F_z = 0$$
$$2(z+1) = -2\lambda z$$
$$z+1 = -\lambda z.$$
$$\boxed{z = \frac{-1}{1+\lambda}.} \rightarrow ③.$$

From ① , ② , ③

$$x = \frac{1}{1+\lambda}, \quad y = \frac{2}{1+\lambda}, \quad z = \frac{-1}{1+\lambda}.$$

$$\boxed{y = 2x}, \quad \boxed{z = -x}$$

Given condition $x^2+y^2+z^2 = 24.$

$$\therefore \; x^2 + (2x)^2 + (-x)^2 = 24.$$

$$x^2 + 4x^2 + x^2 = 24.$$

$$6x^2 = 24.$$

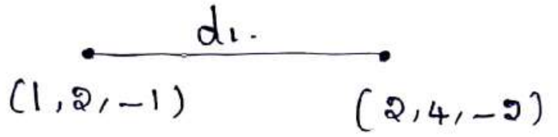
$$x^2 = 4. \Rightarrow x = \pm 2.$$

When $x=2$ $\boxed{\boxed{x=2} \; \boxed{y=4} \; \boxed{z=-2}}$

When $x=-2$, $\boxed{y=-4 \;\; z=2 \;\; x=-2}$

$\therefore$ Points are (x, y, z)

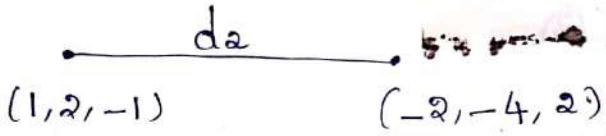
$(2, 4, -2)$, $(-2, -4, 2)$.



$$d_1 = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2}$$

$$= \sqrt{1+4+1}$$

$$d_1 = \sqrt{6}.$$



$$d_2 = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2}$$

$$= \sqrt{9 + 36 + 9}$$

$$d_2 = \sqrt{54}.$$

∴ Longest distance $= \sqrt{54}$

Shortest distance $= \sqrt{6}$.

Unit - IV

Integration.

Formula

$f(x)$	$\int f(x).\,dx$
1) x^n	$\int x^n dx = \frac{x^{n+1}}{n+1}$
2) $\cos x$	$\int \cos x.\,dx = \sin x$
3) $\sin x$	$\int \sin x\, dx = -\cos x$
4) $\int \frac{1}{x}\, dx$	$\log x$
5) $\int \frac{f'(x)}{f(x)}.\,dx$	$\log\left[f(x)\right]$
6) $\int e^x\, dx$	e^x
7) $\int e^{ax}$	$\frac{e^{ax}}{a}$
8) $\int \frac{1}{\sqrt{1-x^2}}\, dx$	$\sin^{-1}x$
9) $\int \frac{-1}{\sqrt{1-x^2}}\, dx$	$\cos^{-1}x$
10) $\int \frac{1}{1+x^2}\, dx$	$\tan^{-1}(x)$.

Problems

① Evaluate $\int \frac{x^3+2x+1}{x^4}\,dx$.

Soln:-

$$\int \frac{x^3+2x+1}{x^4}\,dx = \int \frac{x^3}{x^4}\,dx + 2\int \frac{x}{x^4}\,dx + \int \frac{1}{x^4}\,dx$$

$$= \int \frac{1}{x}\,dx + 2\int x^{-3}\,dx + \int x^{-4}\,dx$$

$$= \log x + 2\left(\frac{x^{-3+1}}{-2}\right) + \left(\frac{x^{-3}}{-3}\right) + C$$

$$= \log x - \frac{1}{x^2} - \frac{1}{3x^3} + C.$$

② Evaluate $\int (x^2+1)(x+3)\,dx$

Soln:-

$$\int (x^2+1)(x+3)\,dx = \int (x^3 + 3x^2 + x + 3).\,dx$$

$$= \frac{x^4}{4} + \frac{3x^3}{3} + \frac{x^2}{2} + 3x. + C$$

$$= \frac{x^4}{4} + x^3 + x^2/2 + 3x + C.$$

③ Evaluate $\int \tan x \, . dx$.

Soln:-

$$\int \tan x . \, dx = \int \frac{\sin x}{\cos x} dx$$

$$= -1 \int \frac{-\sin x}{\cos x} . \, dx$$

$$= -1 \times \log(\cos x)$$

$$= \log(\cos x)^{-1}$$

$$= \log . (\sec x).$$

④ Evaluate $\int \sqrt{1 + \sin 2x} \, . \, dx.$

Soln

$$\int \sqrt{1+\sin 2x} \, dx = \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x} \, dx$$

$$= \int \sqrt{(\sin x + \cos x)^2} \, dx$$

$$= \int (\sin x + \cos x) \, dx$$

$$= -\cos x + \sin x + C.$$

Properties of Integral

1) $\int_a^b f(x).\,dx = \int_a^b f(t).\,dt$

2) $\int_a^b f(x).\,dx = -\int_b^a f(x).\,dx.$

3) $\int_a^c f(x)\,dx + \int_c^b f(x).\,dx = \int_a^b f(x).\,dx$ where $a < c < b$

4) $\int_0^a f(x).\,dx = \int_0^a f(a-x).\,dx$

5) $\int_a^b \left[f(x) + g(x)\right] dx = \int_a^b f(x).\,dx + \int_a^b g(x).\,dx$

6) $\int_0^{2a} f(x).\,dx = \int_0^a f(x).\,dx + \int_0^a f(2a-x).\,dx.$

7) (i) $\int_0^{2a} f(x).\,dx = 2\int_0^a f(x).\,dx$ iff $f(2a-x) = f(x).$

(ii) $\int_0^{2a} f(x).\,dx = 0$ iff $f(2a-x) = -f(x).$

8) (i) If $f(x)$ is even, $\int_{-a}^{a} f(x).\,dx = 2\int_0^a f(x).\,dx$

(ii) If $f(x)$ is odd, $\int_{-a}^{a} f(x).\,dx = 0$

Problems

① Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} . dx$.

Soln:-

Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} . dx . \longrightarrow ①$

$f(x) = \frac{\sin x}{\sin x + \cos x}$, $a = \pi/2$.

$$f(a-x) = f\left(\pi/2 - x\right) = \frac{\sin\left(\pi/2 - x\right)}{\sin\left(\pi/2 - x\right) + \cos\left(\pi/2 - x\right)}$$

$$= \frac{\cos x}{\cos x + \sin x} .$$

We know that

$$\int_0^a f(x) . dx = \int_0^a f(a-x) . dx .$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} . dx \longrightarrow ②$$

① + ②

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} . dx$$

$$= \int_0^{\pi/2} dx$$

$$= \left[x\right]_0^{\pi/2} .$$

$$2I = \pi/2.$$

$$\therefore \boxed{I = \pi/4}$$

② Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx.$

Soln

Let $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \longrightarrow ①$

$$f(x) = \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}, \quad a = \pi/2.$$

$$\therefore f(a-x) = \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}$$

We know that

$$\int_0^a f(x).\, dx = \int_0^a f(a-x).\, dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}.\, dx. \longrightarrow ②$$

① + ②

$$\Rightarrow \quad 2I = \int_0^{\pi/2} dx = \pi/2.$$

$$\therefore \boxed{I = \pi/4}$$

③ Evaluate $\int_0^{\pi/2} \log(\tan x)\, dx$.

Soln:-

$$I = \int_0^{\pi/2} \log(\tan x).\, dx. \longrightarrow ①$$

$f(x) = \log(\tan x)$, $a = \pi/2$

$$f(a-x) = \log\left(\tan(\pi/2 - x)\right)$$

$$= \log.(\cot x).$$

we know that

$$\int_0^a f(x).\, dx = \int_0^a f(a-x).\, dx.$$

$$\int_0^{\pi/2} \log(\tan x).\, dx = \int_0^{\pi/2} \log(\cot x).\, dx.$$

$$\therefore \quad I = \int_0^{\pi/2} \log(\cot x).\, dx. \longrightarrow ②$$

① + ②

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \left[\log(\tan x) + \log(\cot x).\right] dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\tan x}{\tan x}\right) dx$$

$$= \int_0^{\pi/2} \log 1 \, dx$$

$$2I = 0. \quad \Rightarrow \boxed{I = 0}$$

④ Evaluate

$$\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}} \, dx.$$

Soln

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}} \, dx \longrightarrow ①$$

$$f(x) = \frac{1}{1+\sqrt{\tan x}}, \quad a = \pi/6, \; b = \pi/3.$$

$$a+b = \pi/2$$

We know that

$$\int_a^b f(x).\,dx = \int_a^b f(a+b-x).\,dx.$$

$$\therefore f(a+b-x) = \frac{1}{1+\sqrt{\cot x}}.$$

$$\therefore I = \int_a^b \frac{1}{1+\sqrt{\cot x}} \, dx \longrightarrow ②$$

① + ②

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} \left(\frac{1}{1+\sqrt{\tan x}}\right) + \left(\frac{1}{1+\sqrt{\cot x}}\right) . dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} \, dx$$

$$= \left[x\right]_{\pi/6}^{\pi/3} = \pi/\cancel{12}\,6.$$

$$\therefore \boxed{I = \pi/12}$$

⑤ Evaluate $\int_{-1}^{1} \log\left(\frac{4-x}{4+x}\right) dx.$

Soln :-

$$f(x) = \log\left(\frac{4-x}{4+x}\right) dx.$$

$$f(-x) = \log\left(\frac{4+x}{4-x}\right)$$

$$= \log\left(\frac{4-x}{4+x}\right)^{-1}$$

$$= -1 \times f(x)$$

∴ $f(x)$ is odd function

$$\therefore \int_{-1}^{1} \log\left(\frac{4-x}{4+x}\right) dx = 0.$$

⑥ Evaluate $\int_{-\pi/2}^{\pi/2} \sin^{199} x \, . \, dx.$

Soln

$$f(x) = \sin^{199} x \; dx.$$

$$f(-x) = \sin^{199}(-x)$$

$$= \underbrace{\sin(-x) \times \sin(-x) \times \dots \times \sin(-x)}_{199 \text{ times}}$$

$$= -\sin x \times -\sin x \times \dots \times -\sin x$$

$$= -\sin^{199} x$$

$$= -f(x).$$

$\therefore$ f(x) is odd function

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \sin^{199} x \, dx = 0.$$

Integration by Substitution :-

Problems :-

① Evaluate $\int (ax+b)^n \, dx$.

Soln :- Let $I = \int (ax+b)^n \, dx$.

$u = ax+b$

$\frac{du}{dx} = a$

$du = a\,dx$.

$\therefore dx = \frac{du}{a}$

$$\int (ax+b)^n \, dx = \int u^n \frac{du}{a}$$

$$= \frac{1}{a} \int u^n \, du$$

$$= \frac{1}{a}\left(\frac{u^{n+1}}{n+1}\right)$$

$$= \frac{1}{a}\left(\frac{(ax+b)^{n+1}}{n+1}\right).$$

② Evaluate $\int e^{\tan^{-1}x} \left(\frac{1+x+x^2}{1+x^2} \right) dx$.

Soln :-

Let $I = \int e^{\tan^{-1}x} \left(\frac{1+x+x^2}{1+x^2} \right) dx$.

$= \int e^{\tan^{-1}x} (1+x+x^2) \times \frac{1}{1+x^2} . dx$.

Put $u = \tan^{-1}x$, $\Rightarrow x = \tan u$.

$\frac{du}{dx} = \frac{1}{1+x^2}$.

$du = \frac{dx}{1+x^2}$.

$\therefore \quad I = \int e^u (1 + \tan u + \tan^2 u) \; du$

$= \int e^u (\sec^2 u + \tan u). du$

$I = \int e^u \sec^2 u \, du + \int e^u \tan u . du \longrightarrow ①$

Put $F = e^u \tan u$.

$\frac{dF}{du} = e^u \tan u + e^u \sec^2 u. \longrightarrow ②$

Sub ② in ①

$I = \int \frac{dF}{du} = F + c$.

$= e^u \tan u + C$.

w) $I = e^{\tan^{-1}x} \tan(\tan^{-1}x) + C$

$= e^{\tan^{-1}x} \times x + C$

$\therefore I = x \cdot e^{\tan^{-1}x} + C.$

③ Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx.$

Soln:-

Let $I = \int \frac{\sin\sqrt{x}}{\sqrt{x}} dx.$

Put $\sqrt{x} = u.$

$u = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}.$

$\therefore 2\,du = \frac{dx}{\sqrt{x}}.$

$\therefore I = \int \sin u \times 2\,du$

$= 2\int \sin u \cdot du$

$= 2\left[-\cos u\right] + C$

$= -2\cos u + C$

$= -2\cos\sqrt{x} + C.$

Integration by Parts :-

$\int u\,dv$ is known as integration by parts.

There are two formulas for $\int u\,dv$.

1) $\int u\,dv = uv - \int v\,du$

2) Bernoulli Formula:

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + u^{iv}v_4 - \cdots\cdots$$

Here $u', u'', u''', \ldots$ are successive differentiations of u.
$v_1, v_2, v_3, \ldots$ are successive integrations of v.

Note Take u in the following order.

ⓤ Take u from up to down $\downarrow$

I	Inverse	$\sin^{-1}x, \cos^{-1}x, \ldots$
L	Logarithmic	$\log x, \log 2x, \ldots$
A	Algebraic	$x, x^2, (ax+b)^n, \ldots$
T	Trigonometric	$\sin x, \cos x, \sin 2x, \cos 2x, \ldots$
E	Exponential	$e^x, e^{ax}, \ldots$

Problems.

① Evaluate $\int e^{ax}\cos bx\,dx$. using integration by parts [A.U. Jan - 2018]

Soln
Let $I = \int e^{ax}\cos bx\,dx$.

$\int u\,dv = uv - \int v\,du$.

Take $u = \cos bx$ | $dv = e^{ax}dx$

$\frac{du}{dx} = -b\sin bx$. | $\int dv = \int e^{ax}dx$.

$du = -b\sin bx\,dx$ | $v = \frac{e^{ax}}{a}$

$$\therefore\ I = \cos bx\cdot\frac{e^{ax}}{a} + \int\frac{e^{ax}}{a}\,b\sin bx\,dx.$$

$$I. = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx \longrightarrow ①$$

Take $I_1 = \int e^{ax} \sin bx \; dx$.

$$\therefore \quad I = \frac{e^{ax}}{a} \cos bx + \frac{b}{a} I_1 \longrightarrow ② \quad (\text{using } ①)$$

TO find I_1

$$I_1 = \int e^{ax} \sin bx \, . dx$$

$u = \sin bx$ | $dv = e^{ax}$.

$du = b \cos bx \, dx$ | $v = \frac{e^{ax}}{a.}$

$$I_1 = \frac{e^{ax}}{a} \sin bx - \int \frac{e^{ax}}{a} b \cos bx \, . dx$$

$$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a} \int e^{ax} \cos bx \, . dx$$

$$I_1 = \frac{e^{ax}}{a} \sin bx - \frac{b}{a} I . \longrightarrow ③.$$

Sub ③ in ②.

$$I = \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \left(\frac{e^{ax}}{a} \sin bx - \frac{b}{a} I \right).$$

$$I = \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I.$$

$$\therefore \quad I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a} \left[\cos bx + \frac{b}{a} \sin bx \right]$$

$$\therefore \quad I \left(1 + \frac{b^2}{a^2} \right) = \frac{e^{ax}}{a^2} \left[a \cos bx + b \sin bx \right]$$

$$\therefore \quad I = \left(\frac{a^2}{a^2+b^2} \right) \frac{e^{ax}}{a^2} \left[a \cos bx + b \sin bx \right]$$

$$I. = \frac{e^{ax}}{a^2+b^2} \left[a \cos bx + b \sin bx \right] + C.$$

② Evaluate $\int_0^\infty e^{-ax} \sin bx \, dx$ $(a>0)$ using integration by parts. [A.U. April - 2018]

Soln:

Let $I = \int_0^\infty e^{-ax} \sin bx . dx.$

Take $u = \sin bx$ | $dv = e^{-ax}.$

$du = b \cos bx \, dx$ | $v = \frac{e^{-ax}}{-a}.$

$$\int_a^b u dv = \left\{uv\right\}_a^b - \int_a^b v du.$$

$$\therefore \int_0^\infty e^{-ax} \sin bx \, dx = \left\{\frac{\sin bx . e^{-ax}}{-a}\right\}_{x=0}^{x=\infty} - \int_0^\infty \frac{e^{-ax}}{-a} b \cos bx . dx$$

$$= \left\{0-0\right\} + \int_0^\infty \frac{b}{a} e^{-ax} \cos bx . dx.$$

$$I = \frac{b}{a} \int_0^\infty e^{-ax} \cos bx . dx.$$

Take $I_1 = \int_0^\infty e^{-ax} \cos bx . dx.$

$$\therefore I = \frac{b}{a} I_1 , \longrightarrow ①$$

Consider $I_1 = \int_0^\infty e^{-ax} \cos bx . dx.$

Take $u = \cos bx$ | $dv = e^{-ax} dx$

$du = -b \sin bx . dx$ | $v = \frac{e^{-ax}}{-a}$

$$I_1 = \left\{\frac{\cos bx \, e^{-ax}}{-a}\right\}_{x=0}^{x=\infty} - \int_0^\infty \frac{e^{-ax}}{-a} (-b \sin bx) dx.$$

$$= \left\{0 + \frac{1}{a}\right\} - \int_0^\infty \frac{b}{a} \, e^{-ax} \sin bx \, dx$$

$$I_1 = \frac{1}{a} - \frac{b}{a} \int_0^\infty e^{-ax} \sin bx \, dx$$

$$I_1 = \frac{1}{a} - \frac{b}{a} I \longrightarrow ②$$

Sub ② in ①

$$I = \frac{b}{a}\left(\frac{1}{a} - \frac{b}{a} I\right)$$

$$I = \frac{b}{a^2} - \frac{b^2}{a^2} I.$$

$$I + \frac{b^2}{a^2} I = \frac{b}{a^2}.$$

$$I\left(1 + \frac{b^2}{a^2}\right) = \frac{b}{a^2}$$

$$I\left(\frac{a^2+b^2}{a^2}\right) = \frac{b}{a^2}$$

$$I = \frac{b}{a^2} \times \frac{a^2}{(a^2+b^2)}$$

$$\boxed{I = \frac{b}{a^2+b^2}}$$

③ Using integration by parts, evaluate $\int \frac{(\ln x)^2}{x^2} \, dx.$

[A.U. Jan-2019]

Soln:- Let $I = \int \frac{(\ln x)^2}{x^2} \, dx$

(Note $\ln x = \log x.$)

$$\therefore I = \int \frac{(\log x)^2}{x^2} \, dx.$$

$u = (\log x)^2$ | $dv = \frac{dx}{x^2} = x^{-2}dx$

$du = 2\log x \times \frac{1}{x}$ | $v = \int x^{-2}dx$

$= \frac{x^{-1}}{-1}$.

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

$$\therefore \quad I = (\log x)^2 \left(x^{-1}/_{-1}\right) - \int \frac{x^{-1}}{-1} \times 2\log x \times {}^1/_x.$$

$$= -\frac{(\log x)^2}{x} + 2\int \frac{\log x}{x^2}\,dx.$$

$$I = -\frac{(\log x)^2}{x} + 2I_1 \longrightarrow ①$$

where $I_1 = \int \frac{\log x}{x^2}\,dx$.

Consider $I_1 = \int \frac{\log x}{x^2}\,dx$.

$u = \log x$ | $dv = \frac{1}{x^2}\,dx$.

$du = \frac{1}{x}\,dx$ | $v = \frac{x^{-1}}{-1}\,dx. = -{}^1/_x\,dx.$

$$\therefore I_1 = \log x\left(-{}^1/_x\right) - \int -{}^1/_x \cdot \left({}^1/_x\right)dx$$

$$= -\frac{\log x}{x} + \int {}^1/_{x^2}\,dx$$

$$I_1 = -\frac{\log x}{x} + \left(-{}^1/_x\right) \longrightarrow ②$$

Sub ② in ①

$$I = -\frac{(\log x)^2}{x} + 2\left(-\frac{\log x}{x} - {}^1/_x\right) + C.$$

Ⓐ Establish a reduction formula for $I_n = \int \sin^n x \, dx$.

Hence find, $\int_0^{\pi/2} \sin^n x \, dx$. [A.U. Jan – 2019 – 10 mark]

Soln:-

$$I_n = \int \sin^n x \, . dx$$

$$I_n = \int \sin^{n-1+1} x \cdot dx$$

$$= \int \sin^{n-1} x . \sin x . dx.$$

Take $u = \sin^{n-1} x$ | $dv = \sin x \, dx$

$u = (\sin x)^{n-1}$ | $v = -\cos x . ~~dx~~ .$

$du = (n-1) \sin^{n-2} x \cdot \cos x$

$$\boxed{\int u dv = uv - \int v du}$$

$$\therefore I_n = -\sin^{n-1} x . \cos x - \int -\cos x . (n-1) \sin^{n-2} x \cos x \, dx$$

$$= -\sin^{n-1} x \cos x + \int (n-1) \cos^2 x . \sin^{n-2} x . dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int (1-\sin^2 x) \sin^{n-2} x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \left[\int \sin^{n-2} x \, dx - \int \sin^n x dx \right]$$

$$I_n . = -\sin^{n-1} x \cos x + (n-1) \left[I_{n-2} - I_n \right]$$

$$I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}.$$

$$n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}.$$

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \left(\frac{n-1}{n}\right) I_{n-2} \text{ is}$$

$\hookrightarrow$ ①.

required reduction formula.

Now from ①

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \left(\frac{-\sin^{n-1} x \cos x}{n} \right)_{x=0}^{x=\pi/2} + \left(\frac{n-1}{n} \right) I_{n-2}.$$

$$\therefore I_n = (0-0) + \left(\frac{n-1}{n} \right) I_{n-2}.$$

$$I_n = \left(\frac{n-1}{n} \right) I_{n-2}.$$

$$I_n = \left(\frac{n-1}{n} \right) \times \left(\frac{n-3}{n-2} \right) \times \cdots \times \frac{2}{3} \times I_1 \quad \text{if } n \text{ is odd.}$$

$$I_1 = \int_0^{\pi/2} \sin x \, dx = \left(-\cos x \right)_{x=0}^{x=\pi/2} = (-0+1) = 1.$$

$$\therefore I_n = \left(\frac{n-1}{n} \right) \times \left(\frac{n-3}{n-2} \right) \times \cdots \times \frac{2}{3} \times 1 \quad \text{if } n \text{ is odd.}$$

Also

$$I_n = \left(\frac{n-1}{n} \right) \times \left(\frac{n-3}{n-2} \right) \times \cdots \times \frac{1}{2} \times I_0.$$

$$I_0 = \int_0^{\pi/2} dx = \pi/2.$$

$$\therefore I_n = \left(\frac{n-1}{n} \right) \times \left(\frac{n-3}{n-2} \right) \times \cdots \times \frac{1}{2} \times \frac{\pi}{2}$$

if n is even.

② Find the reduction formula $\int_0^{\pi/2} \cos^n x \, . dx$.

Soln:-

$$I_n = \int_0^{\pi/2} \cos^n x . dx$$

$$= \int_0^{\pi/2} \cos^{n-1} x . \cos x . dx .$$

$$u = \cos^{n-1} x \qquad dv = \cos x . dx$$

$$du = -(n-1) \cos^{n-2} x . \sin x \qquad v = \sin x . dx .$$

$$\therefore \quad I_n = \Big[\cos^{n-1} x \sin x\Big]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} x . \sin^2 x . dx$$

$$= (n-1) \int_0^{\pi/2} \cos^{n-2} x . (1 - \cos^2 x) . dx$$

$$= (n-1) \left[\int_0^{\pi/2} \cos^{n-2} x . dx - \int_0^{\pi/2} \cos^n x . dx \right]$$

$$= (n-1) \Big[I_{n-2} - I_n \Big] .$$

$$I_n + (n-1) I_n = (n-1) I_{n-2} .$$

$$n . I_n = (n-1) I_{n-2}$$

$$\therefore \boxed{I_n = \frac{n-1}{n} I_{n-2}}$$

Which is required reduction formula.

Case (i) :- Suppose n is odd

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

$$\therefore \quad I_n = \left(\frac{n-1}{n}\right) \times \left(\frac{n-3}{n-2}\right) I_{n-4}$$

$$I_n = \left(\frac{n-1}{n}\right) \times \left(\frac{n-3}{n-2}\right) \times \dots \times \frac{2}{3} I_1.$$

$$I_1 = \int_0^{\pi/2} \cos x \, . \, dx = 1.$$

$$\therefore \quad I_n = \left(\frac{n-1}{n}\right) \times \left(\frac{n-3}{n-2}\right) \times \dots \times \frac{2}{3} \times 1.$$

Case (ii) :- Suppose n is even.

$$I_n = \left(\frac{n-1}{n}\right) \times I_{n-2}$$

$$= \left(\frac{n-1}{n}\right) \times \left(\frac{n-3}{n-2}\right) I_{n-4}$$

$$= \left(\frac{n-1}{n}\right) \times \left(\frac{n-3}{n-2}\right) \times \dots \times \frac{1}{2} I_0 .$$

$$I_0 = \int_0^{\pi/2} \cos^0 x \, . \, dx = \pi/2 .$$

$$\therefore \quad I_n = \left(\frac{n-1}{n}\right) \times \left(\frac{n-3}{n-2}\right) \times \dots \times \frac{1}{2} \times \frac{\pi}{2} .$$

③ Evaluate $\int_0^{\pi/2} \sin^8 x \, . \, dx$.

Soln:-

$$I_n = \int_0^{\pi/2} \sin^n x \, . \, dx.$$

$n = 8$ (even)

$$\therefore \quad I_n = \left(\frac{n-1}{n}\right) \times \left(\frac{n-3}{n-2}\right) \times \dots \times \frac{1}{2} \times \frac{\pi}{2}$$

$$I_8 = \int_0^{\pi/2} \sin^8 x \, dx = \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{35\pi}{256}.$$

④ Evaluate $\int_0^{\pi/2} \sin^5 x \, dx$.

Soln:-

$$I_n = \int_0^{\pi/2} \sin^n x \, . \, dx$$

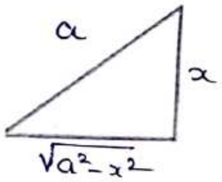
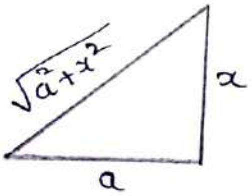
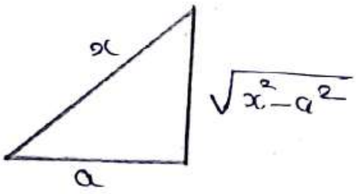
$n = 5$ (odd)

$$\therefore \quad I_n = \left(\frac{n-1}{n}\right) \times \left(\frac{n-3}{n-2}\right) \times \dots \times \frac{2}{3} \times 1.$$

$$I_5 = \int_0^{\pi/2} \sin^5 x \, dx = \frac{4}{5} \times \frac{2}{3} \times 1$$

$$= \frac{8}{15}.$$

Trigonometric Substitutions :-

Expression	Substitution	Diagram
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	

Problems

① Evaluate $\int \frac{1}{\sqrt{a^2-x^2}}\, dx$.

Soln

Let $I = \int \frac{1}{\sqrt{a^2-x^2}}\, dx$

$x = a\sin\theta$

$\frac{dx}{d\theta} = a\cos\theta$

$dx = a\cos\theta . d\theta .$

$$I = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} \times a\cos\theta . d\theta .$$

$$= \int \frac{a\cos\theta}{a\cos\theta}\, d\theta$$

$$= \int d\theta$$

$$I = \theta + C$$

We have

$$x = a\sin\theta$$

$$\Rightarrow \frac{x}{a} = \sin\theta$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \quad I = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

② Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\, dx.$

Soln:-

$$I = \int \frac{\sqrt{9-x^2}}{x^2}\, dx.$$

$$\sqrt{9-x^2} = \sqrt{3^2-x^2}$$

$\therefore$ Substitution : $x = 3\sin\theta.$

$$\frac{dx}{d\theta} = 3\cos\theta.$$

$$dx = 3\cos\theta . d\theta.$$

$$I = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} \times 3\cos\theta \, . \, d\theta$$

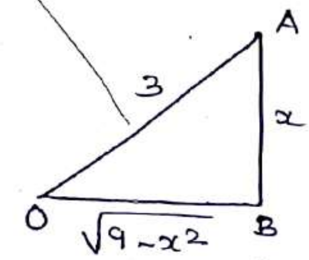
$$= \int \frac{3\cos\theta \times 3\cos\theta}{9\sin^2\theta} \, d\theta$$

$$= \int \cot^2\theta \, . \, d\theta$$

$$= \int (\operatorname{cosec}^2\theta - 1) \, d\theta$$

$$= -\cot\theta - \theta + C \longrightarrow ①$$

Diagram



$$\therefore \quad \cot\theta = \frac{OB}{AB}$$

$$\cot\theta = \frac{\sqrt{9-x^2}}{x} . \longrightarrow ②$$

$$\therefore \quad I = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C .$$

$$\left(\because \ \theta = \sin^{-1}\left(\frac{x}{3}\right)\right)$$

③ Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}} dx$.

Soln:-

$$I = \int \frac{x}{\sqrt{3-2x-x^2}} dx.$$

Consider $\sqrt{3-2x-x^2} = \sqrt{1+3-2x-x^2-1}$

$$= \sqrt{4-(x+1)^2}$$

$$\therefore \quad I = \int \frac{x}{\sqrt{4-(x+1)^2}} dx.$$

Put $x+1 = u$. $\Rightarrow$ $x = u-1$.

$$1 = \frac{du}{dx}$$

$$dx = du.$$

$$\therefore \quad I = \int \frac{u-1}{\sqrt{4-u^2}} du.$$

Substitution: $u = 2\sin\theta$

$$\frac{du}{d\theta} = 2\cos\theta.$$

$$du = 2\cos\theta . d\theta$$

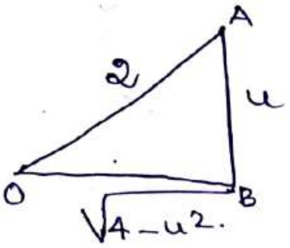
$$\therefore \quad I = \int \frac{(2\sin\theta - 1)}{\sqrt{4-4\sin^2\theta}} \times 2\cos\theta . d\theta$$

$$= \int \frac{(2\sin\theta - 1)}{2\cos\theta} \times 2\cos\theta \; d\theta$$

$$= \int (2\sin\theta - 1)\, d\theta$$

$$= -2\cos\theta - \theta + C \longrightarrow ①$$

Diagram



$$\cos\theta = \frac{OB}{OA}$$

$$= \frac{\sqrt{4-u^2}}{2} \longrightarrow ②$$

∴ Sub ② in ①

$$I = -2\left(\frac{\sqrt{4-u^2}}{2}\right) - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4-u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left[\frac{x+1}{2}\right] + C.$$

④ Evaluate $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$

Soln:-

$$I = \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$$

$$I = \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\left(\sqrt{4x^2+9}\right)^3}\,dx. \longrightarrow ①$$

$$\therefore \sqrt{4x^2+9} = 2\sqrt{x^2+\left(\tfrac{3}{2}\right)^2}$$

Substitution : $x = \frac{3}{2}\tan\theta$

$$\frac{dx}{d\theta} = \frac{3}{2}\sec^2\theta.$$

$$dx = \frac{3}{2}\sec^2\theta . d\theta.$$

x	0	$\frac{3\sqrt{3}}{2}$
θ	0	$\pi/3$

$$\therefore I = \int_0^{\pi/3} \frac{\left(\tfrac{3}{2}\tan\theta\right)^3 \times \frac{3}{2}\sec^2\theta . d\theta}{8 \times \left(\sqrt{\frac{9}{4}\tan^2\theta + \frac{9}{4}}\right)^3}$$

$$= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta \times \frac{3}{2}\sec^2\theta . d\theta}{8 \times \left(\frac{3}{2}\right)^3 \times \sec^3\theta.}$$

$$= \frac{3}{16} \int_{0}^{\pi/3} \frac{\tan^{3}\theta \cdot d\theta}{\sec^{2}\theta}.$$

$$= \frac{3}{16} \int_{0}^{\pi/3} \frac{\tan^{2}\theta \times \tan\theta}{\sec\theta.} \, d\theta$$

$$= \frac{3}{16} \int_{0}^{\pi/3} \frac{(\sec^{2}\theta - 1) \times \tan\theta}{\sec\theta.} \, d\theta.$$

$$= \frac{3}{16} \left[\int_{0}^{\pi/3} \frac{\sec^{2}\theta . \tan\theta}{\sec\theta} \, d\theta - \int_{0}^{\pi/3} \frac{\tan\theta}{\sec\theta} \, d\theta \right]$$

$$= \frac{3}{16} \left[\int_{0}^{\pi/3} \sec\theta . \tan\theta . d\theta - \int_{0}^{\pi/3} \sin\theta \, d\theta \right]$$

$$= \frac{3}{16} \left[\left(\sec\theta \right)_{\theta=0}^{\theta=\pi/3} + \left(\cos\theta \right)_{\theta=0}^{\theta=\pi/3} \right]$$

$$= \frac{3}{16} \left[\sec \pi/3 - \sec 0 + \cos \pi/3 - \cos 0 \right]$$

$$= 3/16 \left[2 - 1 + 1/2 - 1 \right]$$

$$I. = \frac{3}{32}$$

Integrals using Partial Fraction

Partial fraction Rules:-

1) $$\frac{2x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

2) $$\frac{2x}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

3) $$\frac{2x}{(x-1)^2(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}$$

4) $$\frac{2x}{(x^2+4)(x-2)} = \frac{Ax+B}{(x^2+4)} + \frac{C}{x-2}.$$

Problems:-

① Evaluate $\int \frac{dx}{x^2-a^2}$ using partial fraction.

Soln:-

Let $I = \int \frac{dx}{x^2-a^2}$.

Partial fraction:-

$$\frac{1}{(x^2-a^2)} = \frac{1}{(x+a)(x-a)}$$

$$= \frac{A}{(x+a)} + \frac{B}{(x-a)}$$

$$= \frac{A(x-a)+B(x+a)}{x^2-a^2}$$

$\therefore \quad 1 = A(x-a) + B(x+a)$

Put $x = a$	Put $x = -a$
$1 = A(a-a) + B(a+a)$	$1 = A(-a-a) + B(-a+a)$
$1 = 2a \cdot B$	$\boxed{A = -\frac{1}{2a}}$
$\boxed{B = \frac{1}{2a}}$	

$$\therefore \quad \frac{1}{x^2 - a^2} = \frac{-\frac{1}{2a}}{(x+a)} + \frac{\frac{1}{2a}}{(x-a)}.$$

$$\therefore \int \frac{1}{(x^2 - a^2)} dx = \int \frac{-\frac{1}{2a}}{(x+a)} dx + \int \frac{\frac{1}{2a}}{(x-a)} dx$$

$$= -\frac{1}{2a} \int \frac{1}{(x+a)} dx + \frac{1}{2a} \int \frac{1}{(x-a)} dx$$

$$= -\frac{1}{2a} \log(x+a) + \frac{1}{2a} \log(x-a)$$

$$= \frac{1}{2a} \log(x+a)^{-1} + \frac{1}{2a} \log(x-a)$$

$$= \frac{1}{2a} \left[\log\left(\frac{x-a}{x+a}\right) \right] + C.$$

② Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.

Soln:-

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-2x^2+4x+1 \\ & x^4-x^3-x^2+x \\ \hline & x^3-x^2+3x+1 \\ & x^3-x^2-x+1 \\ \hline & 4x \\ \hline \end{array}$$

$\therefore \quad x^4-2x^2+4x+1 = (x^3-x^2-x+1)(x+1)+4x$

$$\Rightarrow \quad \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{4x}{(x^3-x^2-x+1)}. \quad \longrightarrow ①$$

Partial fraction:-

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x+1)(x-1)^2}$$

$$= \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}.$$

$$\therefore \quad 4x = A(x-1)^2 + B(x+1)(x-1) + C(x+1).$$

Put $x = -1$	Put $x = 1$	Equating coeff x^2
$-4 = A(-1-1)^2$	$4 = C(1+1)$	$0 = A + B$
$-4 = 4A.$	$C = \frac{4}{2}$	$B = -A$
$\boxed{A = -1}$	$\boxed{C = 2}$	$\boxed{B = 1}$

$$\therefore \quad \frac{4x}{(x+1)(x-1)^2} = \frac{-1}{(x+1)} + \frac{1}{(x-1)} + \frac{2}{(x-1)^2}$$

∴ From ①

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx = \int \frac{-1}{(x+1)}\, dx + \int \frac{1}{(x+1)}\, dx + \int \frac{2}{(x-1)^2}\, d$$

$$= -1 \log(x+1) + 1 . \log(x-1) . + 2 \int (x-1)^{-2}\, dx$$

$$= \log\left(\frac{x-1}{x+1}\right) + 2\left[\frac{(x-1)^{-2+1}}{-1}\right] + C$$

$$= \log\left(\frac{x-1}{x+1}\right) - 2(x-1)^{-1} + C.$$

③ Evaluate $\int \frac{\sec^2 x \cdot dx}{\tan^2 x + 3\tan x + 2}$.

Soln :-

Let $I = \int \frac{\sec^2 x \cdot dx}{\tan^2 x + 3\tan x + 2}$. $\longrightarrow$ ①

Put $u = \tan x$

$du = \sec^2 x \cdot dx$.

$\therefore$ ① becomes

$$I = \int \frac{1}{(u^2+3u+2)} du.$$

$$= \int \frac{1}{(u+1)(u+2)} du. \longrightarrow ②$$

Partial fraction:-

$$\frac{1}{(u+1)(u+2)} = \frac{A}{(u+1)} + \frac{B}{(u+2)}$$

$$= \frac{A(u+2) + B(u+1)}{(u+1)(u+2)}$$

$\Rightarrow$ $1 = A(u+2) + B(u+1)$

Put $u = -1$	Put $u = -2$
$1 = A(-1+2)$	$1 = B(-2+1)$
$\boxed{A = 1}$	$1 = -B$.
	$\boxed{B = -1}$

$$\therefore \frac{1}{(u+1)(u+2)} = \frac{1}{(u+1)} - \frac{1}{(u+2)}$$

$\therefore$ From ②

$$\int \frac{1}{(u+1)(u+2)} du = \int \frac{1}{(u+1)} du - \int \frac{1}{(u+2)} du$$

$$\underline{I} = \log(u+1) - \log(u+2) + C$$

$$= \log\left(\frac{u+1}{u+2}\right) + C$$

$$\underline{I} = \log\left(\frac{1+\tan x}{2+\tan x}\right) + C.$$

④ $\int \frac{10}{(x-1)(x^2+9)} dx.$

Soln:-

$$I = \int \frac{10}{(x-1)(x^2+9)} dx.$$

Partial fraction:-

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+9)}$$

$$= \frac{A(x^2+9) + (Bx+C)(x-1)}{(x-1)(x^2+9)}$$

$$\therefore \; 10 = A(x^2+9) + (Bx+C)(x-1)$$

Put $x = 1$

$10 = A(1+9)$

$\boxed{A = 1}$

Equating coeff x^2

$0 = A + B$

$-A = B$

$\boxed{B = -1}$

Equating coeff x

$0 = -B + C$

$C = B$

$\boxed{C = -1}$

$$\therefore \; I = \int \frac{1}{(x-1)}\,dx + \int \frac{-x-1}{(x^2+9)}\,dx$$

$$= \int \frac{1}{(x-1)}\,dx - \int \frac{x}{(x^2+9)}\,dx - \int \frac{1}{(x^2+9)}\,dx$$

$$= \log(x-1) - \frac{1}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

($\because$ using formula : $\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$)

⑤ Evaluate $\int \frac{x^2}{(x-1)^3(x-2)}\,dx$.

Soln :-

Let $I = \int \frac{x^2}{(x-1)^3(x-2)}\,dx$

Partial fraction:-

$$\frac{x^2}{(x-1)^3(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-2)}$$

$$= \frac{A(x-1)^2(x-2) + B(x-1)(x-2) + C(x-2) + D(x-1)^3}{(x-1)^3(x-2)}$$

$$\therefore x^2 = A(x-1)^2(x-2) + B(x-1)(x-2) + C(x-2) + D(x-1)^3$$

Put $x = 1$

$1 = -C$

$\boxed{C = -1}$

Put $x = 2$

$4 = D$

$\boxed{D = 4}$

Equating coeff of x^3

$0 = A + D$

$A = -D$

$\boxed{A = -4}$

Equating coeff of x^2

$$1 = -4A + B \cancel{D} - 3D$$

$$B = 4A + 3D + 1$$

$$= 4(-4) + 3(4) + 1$$

$$= -16 + 12 + 1$$

$$\boxed{B = -3}$$

$$\therefore \int \frac{x^2}{(x-1)^3 (x-2)} dx = \int \frac{-4}{(x-1)} dx - \int \frac{3}{(x-1)^2} dx$$

$$- \int \frac{1}{(x-1)^3} dx + \int \frac{4}{(x-2)} dx$$

$$= -4 \log (x-1) - 3 \int (x-1)^{-2} dx - \int (x-1)^{-3} dx$$

$$+ 4 \log (x-2)$$

$$= 4 \log \left(\frac{x-2}{x-1} \right) - 3 \left(\frac{(x-1)^{-1}}{-1} \right) - \left(\frac{(x-1)^{-2}}{-2} \right)$$

$$+ C$$

$$= 4 \log \left(\frac{x-2}{x-1} \right) + \frac{3}{(x-1)} + \frac{1}{2(x-1)^2} + C.$$

UNIT - V

Multiple Integrals

Important Formulae :

1) $\int x^n \, dx = \frac{x^{n+1}}{n+1}$

2) $\int \frac{1}{x} \, dx = \log x$

3) $\int \frac{f'(x)}{f(x)} \, dx = \log \left(f(x)\right)$

4) $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right)$

5) $\int \frac{-1}{\sqrt{a^2-x^2}} \, dx = \cos^{-1}\left(\frac{x}{a}\right)$

6) $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

7) $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

8) $\int \sqrt{x^2-a^2} \, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \log \left(\frac{x+\sqrt{x^2-a^2}}{2}\right)$

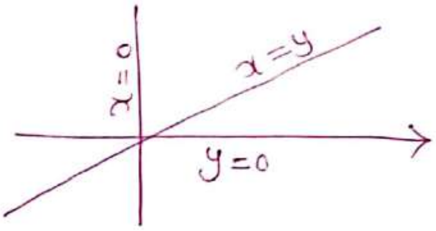
Important Note

Integration Value does not change if we change the variables x into y
(or) x into z

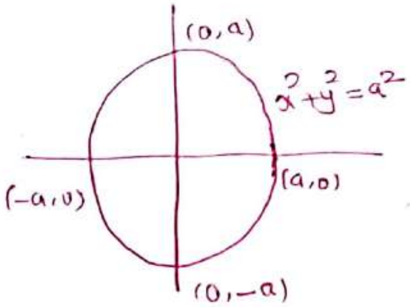
Example $\int x^n dx = \int y^n dy$.

Some Useful Diagrams

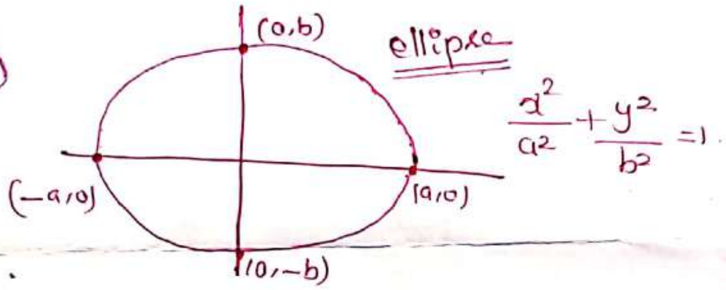
①



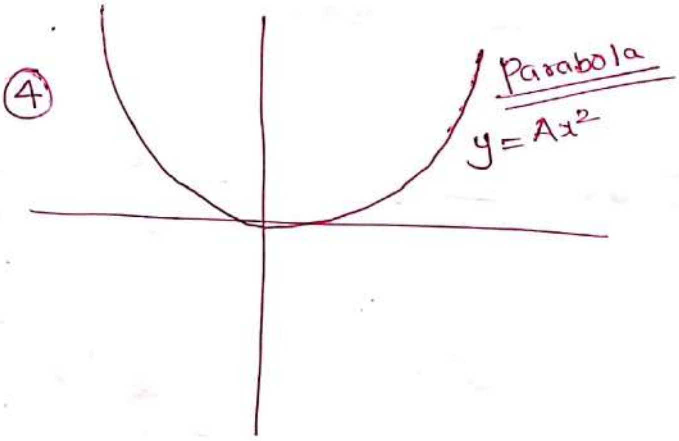
②



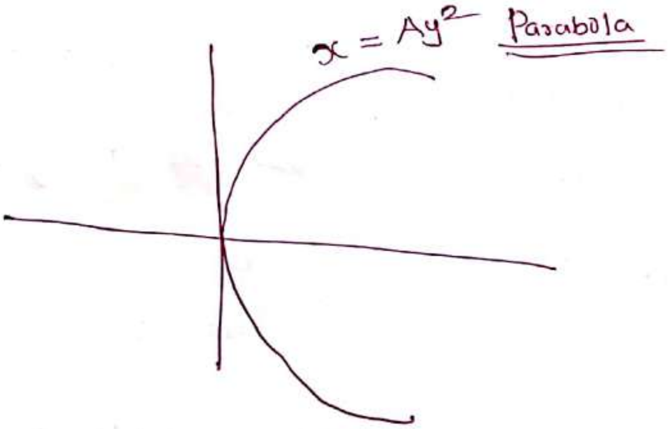
③



④



⑤



Double Integration in Cartesian Co-ordinates :-

Type - I (Limits are constants)

① Evaluate $\int_0^1 \int_1^2 x(x+y)\, dy\, dx$

Soln :-

Let $I = \int_0^1 \int_1^2 x(x+y) \cdot dy\, dx$

$$\therefore \quad I = \int_0^1 \left[\int_1^2 (x^2+xy) \cdot dy \right] dx$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{2} \right]_{y=1}^{y=2} dx$$

$$= \int_0^1 \left[\left\{ x^2(2) + \frac{4x}{2} \right\} - \left\{ x^2 + \frac{x}{2} \right\} \right] dx$$

$$= \int_0^1 \left[2x^2 + 2x - x^2 - \frac{x}{2} \right] dx$$

$$= \int_0^1 \left(x^2 + \frac{3}{2}x \right) \cdot dx$$

$$= \left[\frac{x^3}{3} + \frac{3}{2 \times 2} x^2 \right]_0^1$$

$$= \left[\left\{ \frac{1}{3} + \frac{3}{4} \right\} - \left\{ 0 + 0 \right\} \right]$$

$$= \frac{1}{3} + \frac{3}{4}$$

$$= \frac{4+9}{12}$$

$$\therefore \int_0^1 \int_1^2 x(x+y) dy dx = \frac{13}{12}.$$

② Evaluate $\int_2^a \int_2^b \frac{dx \cdot dy}{xy}$

Soln:-

Let $I = \int_2^a \int_2^b \frac{dx\,dy}{xy}$

$$\therefore I = \int_2^a \left[\int_2^b \frac{dx}{xy} \right] . dy$$

$$= \int_2^a \left[\frac{1}{y} \int_2^b \frac{dx}{x} \right] dy$$

$$= \int_2^a \left[\frac{1}{y} \left(\log x \right)_{x=2}^{x=b} \right] dy$$

$$= \int_2^a \left[\frac{1}{y} \left(\log b - \log 2 \right) \right] dy.$$

$$= \left(\log b - \log 2 \right) \int_2^a \frac{1}{y} dy$$

$$= \log \left(\frac{b}{2} \right) \times \left(\log y \right)_{y=2}^{y=a}$$

$$= \log \left(\frac{b}{2} \right) \times \left(\log a - \log 2 \right)$$

$$= \log \left(\frac{b}{2} \right) \times \log \left(\frac{a}{2} \right)$$

③✓ Evaluate $\int_0^3 \int_0^2 e^{x+y} . dy . dx$.

Soln:-

Let $I = \int_0^3 \int_0^2 e^{x+y} dy . dx.$

$$\therefore I = \int_0^3 \left[\int_0^2 e^{x+y} dy \right] dx.$$

$$= \int_0^3 \left[\int_0^2 e^x . e^y dy \right] dx$$

$$I = \int_0^3 \left[e^x \int_0^2 e^y \, dy \right] dx$$

$$= \int_0^3 \left[e^x \left(e^y \right)_{y=0}^{y=2} \right] dx$$

$$= \int_0^3 \left[e^x \left(e^2 - e^0 \right) \right] dx$$

$$= \int_0^3 \left[e^x \left(e^2 - 1 \right) \right] dx$$

$$= (e^2 - 1) \int_0^3 e^x \, dx$$

$$= (e^2 - 1) \left[e^x \right]_{x=0}^{x=3}$$

$$= (e^2 - 1)(e^3 - e^0)$$

$$= (e^2 - 1)(e^3 - 1)$$

Type-II (Limits are Variable)

If the limits are variable, then check the given problem is in the correct form.

Notes:-

- * Limits of the variable 'x' cannot be interms of 'x'
- * Limits of the variable 'y' cannot be interms of 'y'
- * For correct form arrange 'dx' and 'dy'

④ Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\,(x+y)\,dx\,dy$.

Soln :-

Given $\int_0^1 \int_x^{\sqrt{x}} xy(x+y)\,dx\,dy$.

Here lower limit of $x = x$

Upper limit of $x = \sqrt{x}$.

Limits of the variable 'x' cannot be interms of x.

$$\therefore\ I = \int_0^1 \int_x^{\sqrt{x}} xy(x+y)\,.\,dy\,.\,dx \quad \boxed{\text{correct form}}$$

$$= \int_0^1 \left[\int_x^{\sqrt{x}} xy\,(x+y)\,dy\right] dx$$

$$= \int_0^1 \left[\int_x^{\sqrt{x}} (x^2y + xy^2)\,.\,dy\right] dx.$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{y=x}^{\sqrt{x}=y} dx$$

$$= \int_0^1 \left[\left\{\frac{x^2(\sqrt{x})^2}{2} + \frac{x(\sqrt{x})^3}{3}\right\} - \left\{\frac{x^2(x)^2}{2} + \frac{x(x)^3}{3}\right\}\right] dx$$

$$= \int_0^1 \left[\frac{x^3}{2} + \frac{x\,.\,x^{3/2}}{3} - \frac{x^4}{2} - \frac{x^4}{3}\right] dx$$

$$= \int_0^1 \left[\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{3x^4 - 2x^4}{6}\right] dx$$

$$= \int_0^1 \left[\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{5x^4}{6}\right] dx$$

$$= \left[\frac{x^4}{4\times 2} + \frac{x^{5/2+1}}{(5/2+1)\times 3} - \frac{5x^5}{5\times 6}\right]_0^1$$

$$= \left[\frac{x^4}{8} + \frac{x^{7/2}}{\left(\frac{7}{2}\times 3\right)} - \frac{5x^5}{30}\right]_0^1 = \left[\frac{1}{8} + \frac{1}{21/2} - 5/30\right]$$

$$= \frac{1}{8} + \frac{2}{21} - \frac{1}{6}$$

$$= \frac{(21\times6) + 2(8\times6) - (8\times21)}{8\times21\times6}$$

$$= \frac{126 + 96 - 168}{8\times21\times6}$$

$$= \frac{54}{8\times21\times6}$$

$$= \frac{9}{8\times21}$$

$$= \frac{3}{56} \quad //$$

⑤ Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

Soln :-

$$I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,.\,dx}{1+x^2+y^2}. \quad \left[\text{correct form}\right]$$

$$\therefore\ I = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right] dx.$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{\left(\sqrt{1+x^2}\right)^2+y^2}\right] dx \quad \left(\because \left(\sqrt{1+x^2}\right)^2 = 1+x^2\right)$$

Put $A = \sqrt{1+x^2}$

$$\therefore\ I = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{A^2+y^2}\right] dx$$

Formula $\boxed{\displaystyle\int \frac{dy}{A^2+y^2} = \frac{1}{A}\tan^{-1}\left(\frac{y}{A}\right)}$

$$\therefore\ I = \int_0^1 \left[\frac{1}{A}\tan^{-1}\left(\frac{y}{A}\right)\right]_{y=0}^{y=\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_{y=0}^{y=\sqrt{1+x^2}} dx.$$

$$I = \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\right) - \frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{0}{\sqrt{1+x^2}}\right) \right) dx .$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) - \frac{1}{\sqrt{1+x^2}} \tan^{-1}(0) \right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \times \frac{\pi}{4} - \frac{1}{\sqrt{1+x^2}} \times 0 \right] dx$$

Formula $\boxed{\begin{array}{l} \tan^{-1}(1) = \frac{\pi}{4} \\ \tan^{-1}(0) = 0 . \end{array}}$

$$\therefore I = \int_0^1 \frac{1}{\sqrt{1+x^2}} \times \frac{\pi}{4} . dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx .$$

Formula $\boxed{\int \frac{1}{\sqrt{1+x^2}} dx = \log\left[x+\sqrt{1+x^2}\right]}$

$$\therefore I = \frac{\pi}{4} \left[\log\left(x + \sqrt{1+x^2}\right) \right]_{x=0}^{x=1} .$$

$$= \frac{\pi}{4} \left[\log\left(1+\sqrt{1+1}\right) - \log\left(0+\sqrt{1+0}\right) \right]$$

$$= \frac{\pi}{4} \left[\log\left(1+\sqrt{2}\right) - \log(1) \right]$$

$$I = \frac{\pi}{4} \left[\log(1+\sqrt{2}) \right] \qquad (\because \log 1 = 0)$$

Indicate the region of Integration :-

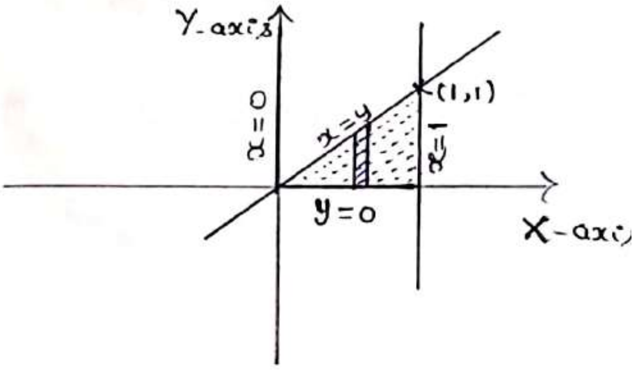
⑥ Sketch roughly the region of integration for $\int_0^1 \int_0^x f(x,y)\,dy\,dx$

Soln :-

Given $\int_0^1 \int_0^x f(x,y) \,.\, dy\, dx.$

Limit Table.

Variable	lower	Upper
x	0	1
y	0	x.



(Strip → ▨ → Drawing Method

In given problem last we have 'dx'

So Draw ▨ perpendicular to x - axis

Suppose last we have 'dy'

Draw ▤ perpendicular to Y - axis).

⑦ Shade the region of entegration $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dx.dy.$

Soln

Let $I = \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$ [correct form]

Limit Table

Var	low	Upp
x	0	a
y	$\sqrt{ax-x^2}$	$\sqrt{a^2-x^2}$

Circle eqn with centre (a,b) and radius 'r' is

$$(x-a)^2 + (y-b)^2 = r^2$$

Lower limit of y

$y = \sqrt{ax - x^2}$

$y^2 = ax - x^2$

$x^2 - ax + y^2 = 0.$

$x^2 - ax + \frac{a^2}{4} + y^2 = \frac{a^2}{4}.$

$(x - a/2)^2 + (y-0)^2 = (a/2)^2$

∴ Lower limit of y is a circle with centre → $(\frac{a}{2}, 0)$

radius → $(\frac{a}{2})$

Similarly, Upper limit of y

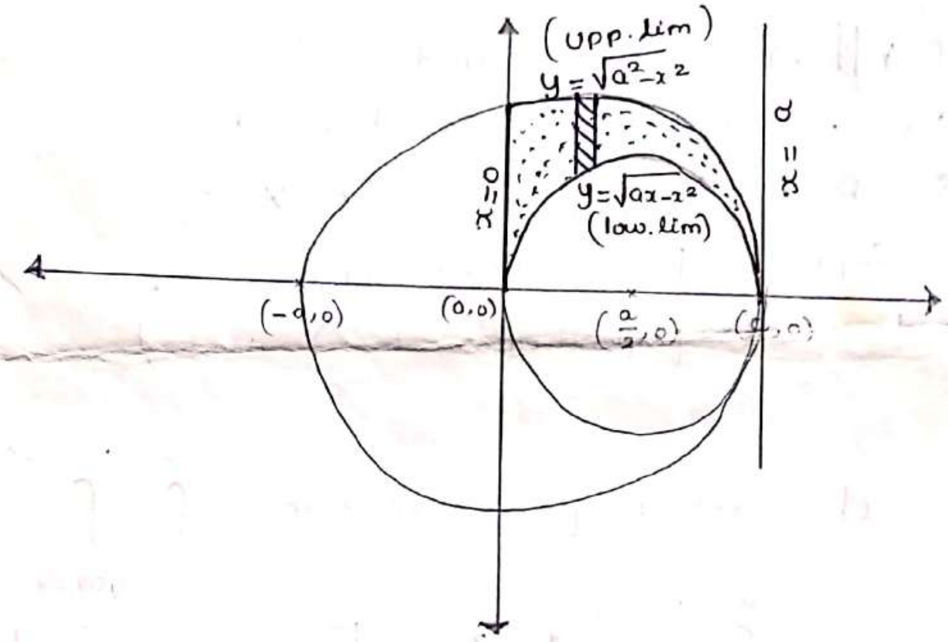
$$y = \sqrt{a^2 - x^2}$$

$$y^2 = a^2 - x^2$$

$$x^2 + y^2 = a^2$$

$$(x-0)^2 + (y-0)^2 = a^2$$

∴ Upper limit of y is a circle with centre → (0,0)
and radius → a.



Double Integration in Polar coordinates :-

① Evaluate $\int_0^{\pi/2} \int_0^{\sin\theta} r\, d\theta\, dr$

Soln

Let $I = \int_0^{\pi/2} \int_0^{\sin\theta} r\, dr\, d\theta$ [correct form]

$$= \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{r=0}^{r=\sin\theta} . d\theta$$

$$I = \int_0^{\pi/2} \left[\frac{\sin^2\theta}{2} - 0 \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin^2\theta \cdot d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\frac{1-\cos 2\theta}{2} \right] d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} (1-\cos 2\theta) . d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{1}{4} \left[\left(\pi/2 - \frac{\sin 2(\frac{\pi}{2})}{2} \right) - (0-0) \right]$$

$$= \frac{1}{4} \left[\pi/2 - 0 \right]$$

$$= \pi/8$$

② Evaluate $\int_0^{\pi} \int_0^{\sin\theta} r \, dr . d\theta .$

Soln :-

$$I = \int_0^{\pi} \left[\int_0^{\sin\theta} r . dr . \right] d\theta$$

$$= \int_0^{\pi} \left[\frac{r^2}{2} \right]_{r=0}^{r=\sin\theta} d\theta$$

$$= \int_0^{\pi} \frac{\sin^2\theta}{2} . d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \sin^2\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \left(\frac{1-\cos 2\theta}{2} \right) . d\theta$$

$$I = \frac{1}{4}\int_0^{\pi}(1-\cos 2\theta).d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - (0-0)\right]$$

$$= \frac{1}{4}\times\pi$$

$$= \frac{\pi}{4}$$

③ Evaluate $\int_0^{\pi}\int_0^{a} r\,dr.d\theta$

Soln :-

Let $I = \int_0^{\pi}\left[\int_0^{a} r\,dr\right]d\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a}d\theta$$

$$= \int_0^{\pi}\frac{a^2}{2}.d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}d\theta$$

$$= \frac{a^2}{2}\left[\theta\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{a^2}{2}(\pi-0)$$

$$= \frac{\pi a^2}{2}.$$

Change Of Order Of Integration

* If limits are constants then change the order of integration by putting 'x' limits for 'x' and by putting 'y' limits for 'y'

① Prove that $\int_0^3 \int_1^2 xy \, dy \, dx = \int_1^2 \int_0^3 xy \, dx \, dy$

Soln

Given. $\int_0^3 \int_1^2 xy . \, dy \, dx$

Var	low	Upp
x	0	3
y	1	2

By changing order of integration

Var	low	Upp
y	1	2
x	0	3

$$\therefore \int_0^3 \int_1^2 xy \, dy \, dx = \int_1^2 \int_0^3 xy \, dx \, dy.$$

Proof By Evaluation :-

Consider $\int_0^3 \int_1^2 xy \, dy \, dx$

$$= \int_0^3 x \left(\frac{y^2}{2}\right)_{y=1}^{y=2} dx$$

$$= \int_0^3 x \left(4/2 - 1/2\right) . dx .$$

$$= \int_0^3 x \left(2 - 1/2\right) . dx$$

$$= 3/2 \int_0^3 x \, dx$$

$$= 3/2 \left[x^2/2\right]_{x=0}^{x=3} = \frac{3}{2} \times \frac{9}{2} = \frac{27}{4}$$

Consider $\int_1^2 \int_0^3 xy \, dx \, dy$

$$= \int_1^2 y \left[x^2/2\right]_0^3 dy$$

$$= \int_1^2 y \left(\frac{9}{2}\right) dy$$

$$= \frac{9}{2} \left(y^2/2\right)_1^2$$

$$= \frac{9}{2} \left(2 - 1/2\right)$$

$$= 9/2 \times 3/2$$

$$= 27/4 .$$

$\therefore$ L.H.S = R.H.S.

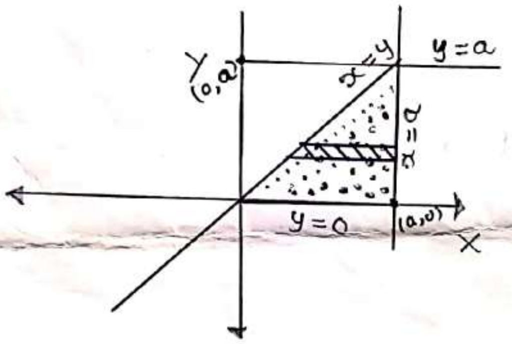
* When the limits are variable the change in the order of integration changes the limits of integration. To find the new limits it is always advisable to draw a rough sketch of the region of integration.

② change the order of integration of $\int_0^a \int_y^a f(x,y).\,dx\,dy$.

Soln :-

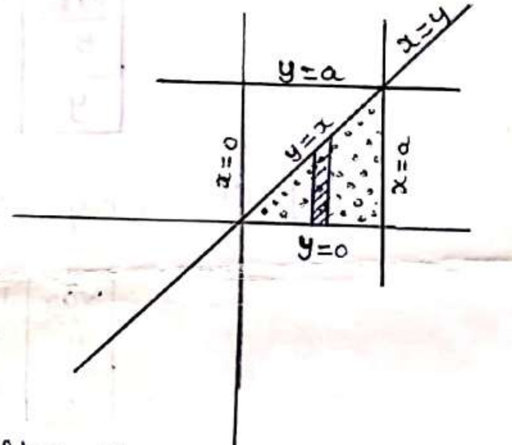
Given $\int_0^a \int_y^a f(x,y).\,dx\,dy$.

Variable	low	Upp
x	y	a
y	0	a



Changing

$\Rightarrow$



After changing

var	low	Upp
y	0	x
x	0	a

∴ By change of order of integration

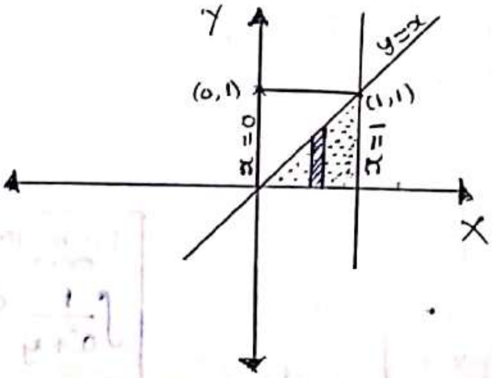
$$\int_0^a \int_y^a f(x,y)\,dx\,dy = \int_0^a \int_0^x f(x,y)\,dy\,dx.$$

② Change the order of integration $\int_0^1 \int_0^x f(x,y)\, dy\, dx$.

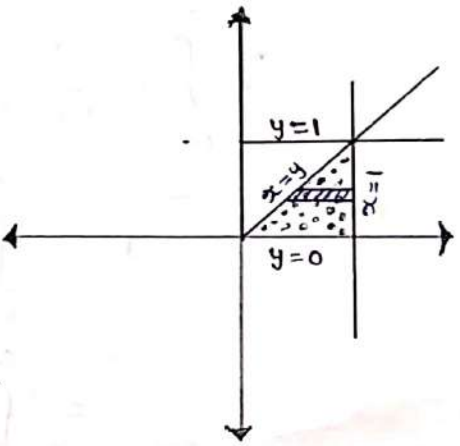
Soln :-

Given $\int_0^1 \int_0^x f(x,y).\, dy.\, dx$

Var	low	Upp
x	0	1
y	0	x



Changing $\Longrightarrow$



After changing

Var	low	Upp
x	y	1
y	0	1

∴ By change of order of integration

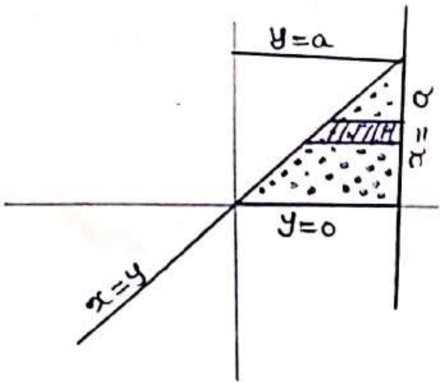
$$\int_0^1 \int_0^x f(x,y).\, dy\, dx = \int_0^1 \int_y^1 f(x,y)\, dx\, dy$$

③ Change the order of $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ and then evaluate

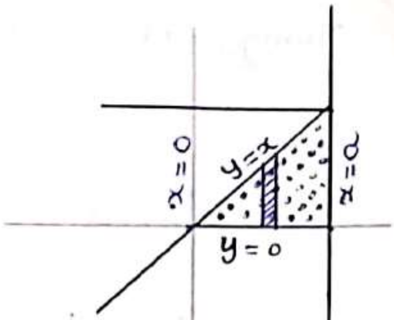
Soln :-

Let $I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$

Var.	low.	Upp
x	y	a
y	0	a



$$\underline{\text{Changing}} \quad \Longrightarrow$$



After changing

Var	low	upp
x	0	a
y	0	x

$$\therefore \int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy = \int_0^a\int_0^x \frac{x}{x^2+y^2}\,dy\,dx$$

$$\underline{I} = \int_0^a \left[\int_0^x \frac{x}{x^2+y^2}\,dy\right] dx$$

Formula

$$\int \frac{1}{a^2+y^2}\,dy = \frac{1}{a}\tan^{-1}\left(\frac{y}{a}\right)$$

$$= \int_0^a \left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right) \times x\right]_{y=0}^{y=x} dx$$

$$= \int_0^a \left[\tan^{-1}\left(\frac{y}{x}\right)\right]_{y=0}^{y=x} dx$$

$$= \int_0^a \left[\tan^{-1}\left(\frac{x}{x}\right) - \tan^{-1}\left(\frac{0}{x}\right)\right] dx$$

$$= \int_0^a \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$\boxed{\begin{array}{l}\tan^{-1}(1) = \frac{\pi}{4}\\ \tan^{-1}(0) = 0\end{array}}$$

$$= \int_0^a \left[\frac{\pi}{4} - 0\right] dx$$

$$= \frac{\pi}{4}\int_0^a dx$$

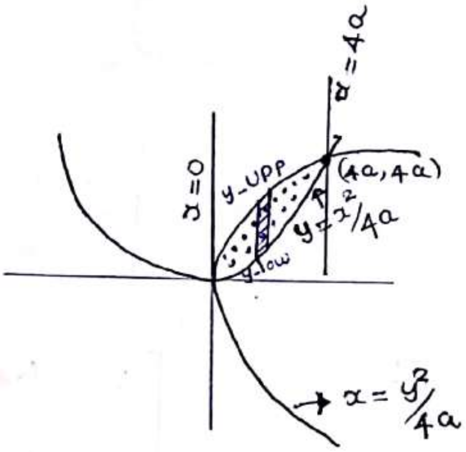
$$= \frac{\pi}{4}\left[x\right]_0^a$$

$$\therefore \underline{I} = \frac{\pi}{4} \times a$$

④ change the order of integration and hence evaluate $\int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\; dy\, dx$

Soln:-

Let $I = \int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy .\, dy\, dx$



Var	low	upp
x	0	$4a$
y	$x^2/4a$	$2\sqrt{ax}$

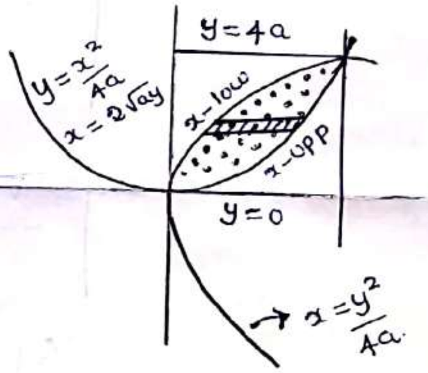
$y = \frac{x^2}{4a}$ → parabola

$y = 2\sqrt{ax}$

$y^2 = 4ax$

$x = \frac{y^2}{4a}$ → parabola.

Changing ⇓



∴ After changing

var	low	upp
x	$y^2/4a$	$2\sqrt{ay}$
y	0	$4a$

∴ By change of order of integration

$$\int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\; dy\, dx = \int_{0}^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\; dx\, dy$$

$$I = \int_{0}^{4a}\left[\int_{y^2/4a}^{2\sqrt{ay}} xy\; dx\right] dy$$

$$= \int_{0}^{4a}\left[y\left(\frac{x^2}{2}\right)_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}}\right] dy$$

$$= \int_{0}^{4a}\left[y\left(\frac{4ay}{2} - \frac{y^4}{16a^2\times 2}\right)\right] dy$$

$$I = \int_0^{4a} y\left(2ay - \frac{y^4}{32a^2}\right)dy$$

$$= \int_0^{4a}\left(2ay^2 - \frac{y^5}{32a^2}\right).dy$$

$$= \left[\frac{2ay^3}{3} - \frac{y^6}{6\times32a^2}\right]_{y=0}^{y=4a}$$

$$= \left[\frac{2a\times64a^3}{3} - \frac{64a^3\times64a^3}{6\times32a^2}\right]$$

$$= \left[\frac{2\times64a^4}{3} - \frac{2a^3\times64a^3}{6a^2}\right]$$

$$= \left[\frac{2\times64a^4}{3} - \frac{64a^4}{3}\right]$$

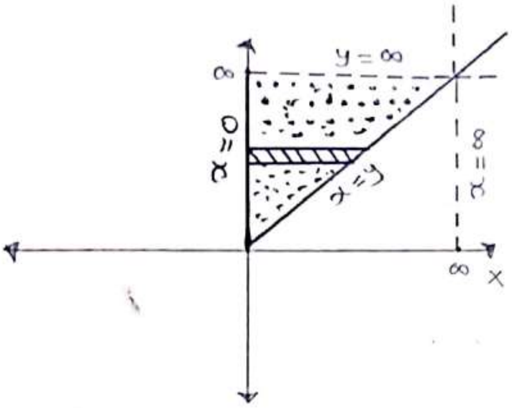
$$= \frac{64a^4}{3}\left[2-1\right]$$

$$I = \frac{64a^4}{3}$$

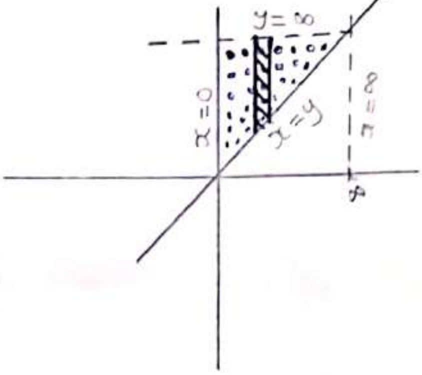
⑤ Change the order of integration in $\int_0^\infty\int_0^y ye^{-y^2/x}dx\,dy$ and hence evaluate it.

Soln :- $I = \int_0^\infty\int_0^y ye^{-y^2/x}\,dx\,dy$

Var	low	UPP
x	0	y
y	0	∞



Changing $\Longrightarrow$



After changing

Var	low	upp
x	0	∞
y	x	∞

$\therefore$ By change of order of integration

$$\int_0^\infty \int_0^y y e^{-y^2/x}\, dx\, dy = \int_0^\infty \int_x^\infty y e^{-y^2/x}\, dy\, dx$$

$$I = \int_0^\infty \left[\int_x^\infty y e^{-y^2/x} dy \right] dx. \longrightarrow ①$$

consider $\displaystyle\int_x^\infty y e^{-y^2/x}\, dy$

Put $y^2 = z.$ $\Rightarrow \sqrt{z} = \sqrt{y^2} \Rightarrow \sqrt{z} = y$

$\therefore$ Diff w.r.t 'z'

$$2y \cdot \frac{dy}{dz} = 1.$$

$$\therefore\ dy = \frac{dz}{2y}$$

$$dy = \frac{dz}{2\sqrt{z}}.$$

$y \to z$		
Var	low	upp
y	x	∞
z	x^2	∞

$y^2 = z$
Put $y = x$
$x^2 = z$
Put $y = \infty$
$\infty^2 = z$
$\infty = z$

$$\therefore \int_x^\infty y e^{-y^2/x}\, dy = \int_{x^2}^\infty \sqrt{z} \cdot e^{-z/x} \frac{dz}{2\sqrt{z}}$$

$$= \frac{1}{2} \int_{x^2}^\infty e^{-z/x}\, dz$$

$$= \frac{1}{2}\left[\frac{e^{-z/x}}{-1/x}\right]_{z=x^2}^{z=\infty}$$

$$= \frac{1}{2}\left[-x\cdot e^{-z/x}\right]_{z=x^2}^{z=\infty}$$

$$= \frac{1}{2}\left[-x\ e^{-\infty/x} + x e^{-x^2/x}\right]$$

$$= \frac{1}{2}\left[-x\times 0 + x e^{-x}\right] \qquad \left(\because e^{-\infty} = 0\right)$$

$$\int_x^\infty y e^{-y^2/x}\, dy = \frac{1}{2}\cdot x e^{-x}. \longrightarrow ②$$

∴ Sub ② in ①

$$I = \int_0^\infty \frac{1}{2}\, x e^{-x}\, dx.$$

$u = x \qquad dv = e^{-x}$

$du = dx \qquad v = \frac{e^{-x}}{-1} = -e^{-x}$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$\therefore I = \frac{1}{2}\left[\left(x\times -e^{-x}\right)_0^\infty - \int_0^\infty -e^{-x}\, dx\right]$$

$$= \frac{1}{2}\left[\left(-xe^{-x}\right)_0^\infty + \int_0^\infty e^{-x}\, dx\right]$$

$$= \frac{1}{2}\left[(0-0) + \left[\frac{e^{-x}}{-1}\right]_0^\infty\right]$$

$$= \frac{1}{2}\left[(0-0) + \left(e^{-\infty} + e^{0}\right)\right]$$

$$= \frac{1}{2}\left(e^{0}\right) = \frac{1}{2}(1) = \frac{1}{2} \;//.$$

⑥ change the order of integration and hence evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

Soln:-

Let $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$

var	low	upp
x	0	1
y	x^2	$2-x$

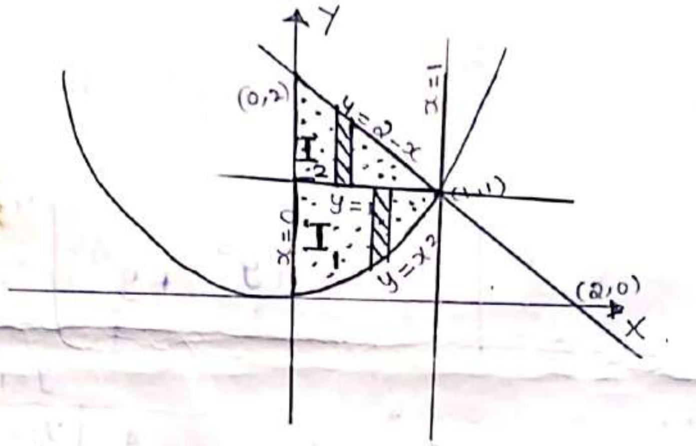
Table for $y = 2-x$

x	0	1	2
y	2	1	0

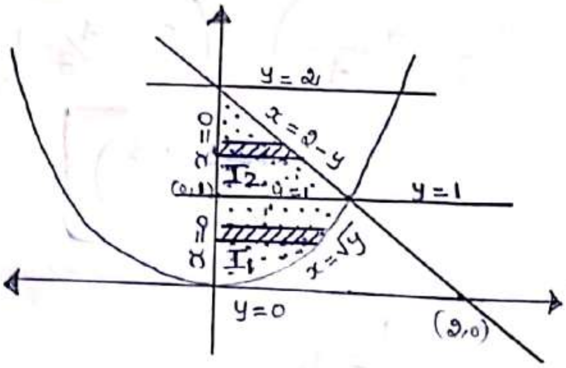
We divide I as

$I = I_1 + I_2$

$I = \int_0^1 \int_{x^2}^{1} xy \, dy \, dx + \int_0^1 \int_1^{2-x} xydyd$



Changing ⇓



After changing

$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy$

$I_2 = \int_1^2 \int_0^{2-y} xy \, dx \, dy.$

∴ By change of order of integration

$I = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy + \int_1^2 \int_0^{2-y} xy \, dx \, dy$

$$I = \int_0^1 \left[y\left(\frac{x^2}{2}\right)_{x=0}^{x=\sqrt{y}} \right] dy + \int_1^2 \left[y \left(\frac{x^2}{2}\right)_{x=0}^{x=2-y}\right] dy$$

$$= \int_0^1 \left[\frac{y}{2}\left(x^2\right)_{x=0}^{x=\sqrt{y}}\right]dy + \int_1^2\left[\frac{y}{2}\left(x^2\right)_{x=0}^{x=2-y}\right]dy$$

$$= \int_0^1 \frac{y}{2}\left(y-0\right)dy + \int_1^2 \frac{y}{2}\left((2-y)^2 - 0\right)dy$$

$$= \int_0^1 \frac{y}{2}\times y \; dy + \int_1^2 \frac{y}{2}\left(4-4y+y^2\right)dy$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 \left(4y-4y^2+y^3\right)dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_{y=0}^{y=1} + \frac{1}{2}\left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_{y=1}^{y=2}$$

$$= \frac{1}{2}\left[\frac{1}{3} - 0\right] + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_{y=1}^{y=2}$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{4\times 8}{3} + \frac{16}{4}\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(12 - \frac{32}{3}\right) - \left(\frac{24-16+3}{12}\right)\right]$$

~~$= \frac{1}{6} + \frac{1}{2}\left[\frac{4}{3} - \frac{32}{3} - \frac{11}{12}\right]$~~

~~$= \frac{1}{6} + \frac{1}{2}\left[\frac{\quad}{12}\right.$~~

$$= \frac{1}{6} + \frac{1}{2}\left[\left(\frac{36-32}{3}\right) - \left(\frac{11}{12}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{4}{3} - \frac{11}{12}\right] = \frac{1}{6} + \frac{1}{2}\left[\frac{16-11}{12}\right] = \frac{1}{6} + \frac{1}{2}\left(\frac{5}{12}\right)$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8} \;/\!/$$

Homework :-

(7) Change the order of integration in $\int_0^a \int_{x^2/a}^{2a-x} xy\, dx\, dy$ and hence evaluate the same.

(Model : Problem (6)
Ans : $\frac{3}{8} a^4$)

Area Enclosed by Plane Curves :-

Formula

Cartesian coordinates } → $\boxed{\text{Area} = \iint dx\, dy \quad \text{(or)} \quad \text{Area} = \iint dy\, .\, dx}$

Polar coordinates } → $\boxed{\text{Area} = \iint r\, dr\, d\theta}$

Problems Based on Cartesian Coordinates :-

(1) Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

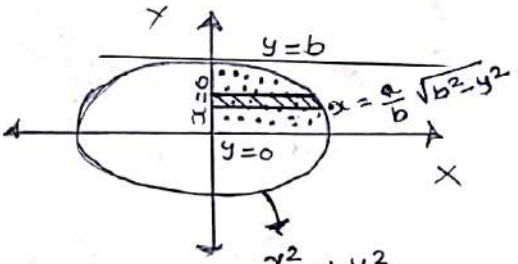
Soln :-

Let I = required Area

Then $I = 4 \times A \longrightarrow (1)$

where A = area of ellipse in 1st Quadrant

Var	low	Upp
x	0	$\frac{a}{b}\sqrt{b^2-y^2}$
y	0	b



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$\therefore \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}.$$

$$\frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2}.$$

$$x^2 = \frac{a^2}{b^2}(b^2 - y^2)$$

$$x = \frac{a}{b}\sqrt{b^2 - y^2}.$$

$$\therefore \quad A = \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dx \, dy$$

Formula

$$\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}$$

$$= \int_0^b \left[x \right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \int_0^b \frac{a}{b}\sqrt{b^2-y^2}\, dy$$

$$= \frac{a}{b}\int_0^b \sqrt{b^2-y^2}\, dy .$$

$$A = \frac{a}{b}\left[\frac{b^2}{2}\sin^{-1}\left(\frac{y}{b}\right) + \frac{y}{2}\sqrt{b^2-y^2} \right]_{y=0}^{y=b}$$

$$= \frac{a}{b}\left[\left(\frac{b^2}{2}\sin^{-1}\left(\frac{b}{b}\right) + \frac{b}{2}\sqrt{b^2-b^2} \right) - \left(\frac{b^2}{2}\sin^{-1}(0) + 0 \right) \right]$$

$$= \frac{a}{b}\left[\frac{b^2}{2}\sin^{-1}(1) + 0 - 0 - 0 \right] \qquad \left(\because \sin^{-1}(0) = 0 \atop \sin^{-1}(1) = \pi/2 \right.$$

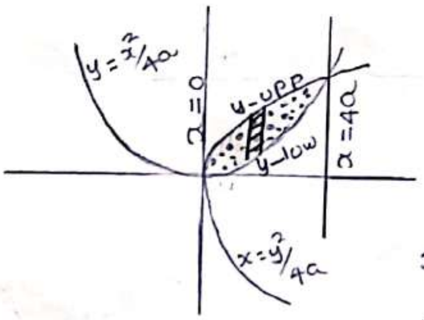
$$= \frac{a}{b} \times \frac{b^2}{2} \times \frac{\pi}{2} = \frac{\pi ab}{4}.$$

$\therefore$ From ① $I = 4 \times \frac{ab \times \pi}{4} = \pi ab.$

② Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Soln Given $y^2 = 4ax \Rightarrow x = y^2/4a$

$x^2 = 4ay \Rightarrow y = x^2/4a.$



Var	low	upp
x	0	$4a$.
y	$x^2/4a$	$2\sqrt{ax}$

$x = y^2/4a$

$\Rightarrow y = 2\sqrt{ax}$

Required area $= \int_{0}^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy . dx$

$$= \int_{0}^{4a} \Big[y \Big]_{y = x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_{0}^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a} \right) dx$$

$$= \int_{0}^{4a} \left(2\sqrt{a}\, x^{1/2} - \frac{x^2}{4a} \right) dx$$

$$= \left[2\sqrt{a}\, \frac{x^{3/2}}{3/2} - \frac{x^3}{3 \times 4a} \right]_{x=0}^{x=4a}$$

$$= \left[2\sqrt{a} \times \frac{2}{3} \times x^{3/2} - \frac{x^3}{12a} \right]_{x=0}^{x=4a}$$

$$= \left[\frac{4\sqrt{a}}{3}\, x^{3/2} - \frac{x^3}{12a} \right]_{x=0}^{x=4a}$$

$$= \left[\frac{4\sqrt{a}}{3} (4a)^{3/2} - \frac{(4a)^3}{12a} \right]$$

$$= \left[\frac{4\sqrt{a} . 4^{3/2} . a^{3/2}}{3} - \frac{4 \times 4 \times 4 \times a^3}{12\, a} \right]$$

$$= \left[\frac{4\sqrt{a} . 4 \times 4^{1/2} \times a^{1} \times a^{1/2}}{3} - \frac{16a^3}{3a} \right]$$

$$= \left[\frac{4 \times \sqrt{a} \times 4 \times 2 \times a \times \sqrt{a}}{3} - \frac{16a^2}{3} \right]$$

$$= \left[\frac{2 \times 16\, a^2}{3} - \frac{16a^2}{3} \right]$$

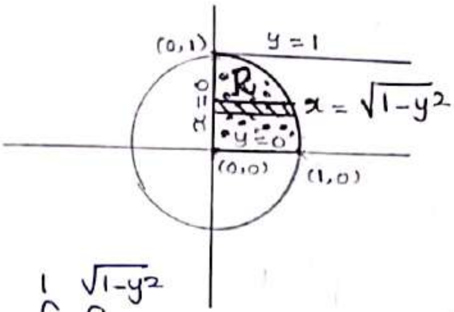
$$= \frac{16a^2}{3} \Big[2 - 1 \Big]$$

Req. Area $= 16a^2/3$

③ Evaluate $\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2+y^2=1$.

Soln

Let $I = \iint_R xy \, dx\, dy$ where R = region of +ve quadrant of the circle $x^2+y^2=1$.



Var	low	upp
x	0	$\sqrt{1-y^2}$
y	0	1

$$\therefore \; I = \int_0^1 \int_0^{\sqrt{1-y^2}} xy \, . \, dx\, dy$$

$$= \int_0^1 \left[\int_0^{\sqrt{1-y^2}} xy \, dx \right] dy$$

$$= \int_0^1 \left[y \left(\frac{x^2}{2} \right)_{x=0}^{x=\sqrt{1-y^2}} \right] dy$$

$$= \int_0^1 \left[\frac{y}{2} \cdot \left(\left(\sqrt{1-y^2}\right)^2 - 0 \right) \right] dy$$

$$= \int_0^1 \left[\frac{y}{2} (1-y^2) \right] dy$$

$$= \frac{1}{2} \int_0^1 (y - y^3)\, dy$$

$$= \frac{1}{2} \left[y^2/2 - y^4/4 \right]_{y=0}^{y=1}$$

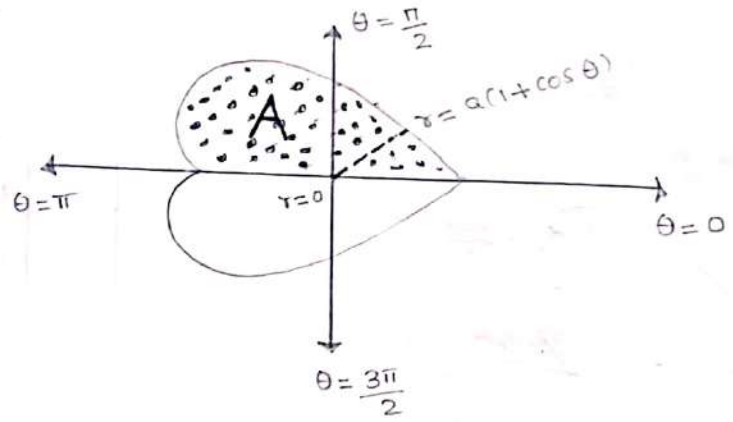
$$= 1/2 \left[\frac{1}{2} - \frac{1}{4} \right]$$

$$= \frac{1}{2} \left[\frac{1}{4} \right]$$

$$= \frac{1}{8} \; //$$

(4) Find using a double integral, the area of the cardiod $r = a(1+\cos\theta)$

Soln:-



In A

θ varies from 0 to π

r varies from 0 to $a(1+\cos\theta$

Required Area = Area of cardiod = 2 x A.

$$= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r.\,dr\,d\theta$$

$$= 2\left[\int_0^{\pi} \left(\frac{r^2}{2}\right)_{r=0}^{r=a(1+\cos\theta)} .\,d\theta\right]$$

$$= \frac{2}{2}\left[\int_0^{\pi} a^2(1+\cos\theta)^2 .\,d\theta\right]$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta).\,d\theta$$

Formula

$$\boxed{\cos^2\theta = \frac{1+\cos2\theta}{2}}$$

$$= a^2\int_0^{\pi}\left[1+2\cos\theta+\left(\frac{1+\cos2\theta}{2}\right)\right]d\theta$$

$$= a^2\int_0^{\pi}\frac{2+4\cos\theta+1+\cos2\theta}{2}\,d\theta.$$

$$= \frac{a^2}{2}\int_0^{\pi}(3+4\cos\theta+\cos2\theta).\,d\theta$$

$$= \frac{a^2}{2}\left[3\theta+4\sin\theta+\frac{\sin2\theta}{2}\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{a^2}{2}\left[\left\{3\pi+4\sin\pi+\frac{\sin2\pi}{2}\right\}-\left\{0+0+0\right\}\right]$$

$$= \frac{a^2}{2}\times 3\pi \qquad (\because \sin\pi = \sin2\pi = 0)$$

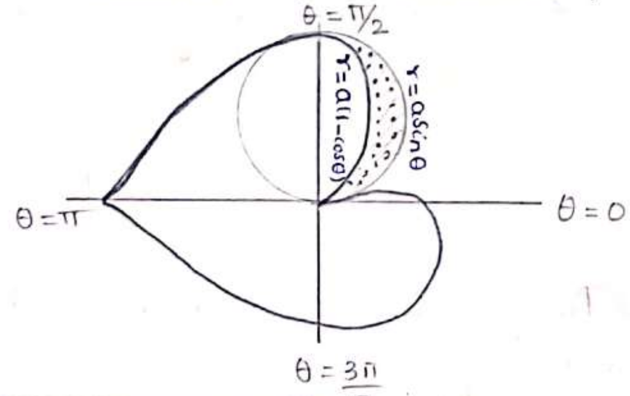
$$= \frac{3\pi a^2}{2} \,//.$$

⑮ Find the area inside the circle $r = a\sin\theta$ and outside the cardiod $r = a(1-\cos\theta)$.

Soln:-

Given : circle $\rightarrow r = a\sin\theta$

cardiod $\rightarrow r = a(1-\cos\theta)$.



	low	upp
r	$a(1-\cos\theta)$	$a\sin\theta$
θ	0	$\frac{\pi}{2}$

$$\boxed{\text{Required area} = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta}$$

$$= \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_{r=a(1-\cos\theta)}^{r=a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[a^2\sin^2\theta - a^2(1-\cos\theta)^2\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(a^2\sin^2\theta - a^2\left[1-2\cos\theta+\cos^2\theta\right]\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[\left(\frac{1-\cos2\theta}{2}\right) - 1 + 2\cos\theta - \left(\frac{1+\cos2\theta}{2}\right)\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[\frac{1-\cos2\theta-2+4\cos\theta-1-\cos2\theta}{2}\right] d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi/2}\left(-2-2\cos2\theta+4\cos\theta\right) d\theta$$

$$= \frac{a^2}{4}\left[-2\theta - \frac{\cancel{2}\sin 2\theta}{\cancel{2}} + 4\sin\theta\right]_{\theta=0}^{\theta=\pi/2}$$

$$= \frac{a^2}{4}\left[\left\{-2\times\frac{\pi}{2} - \sin 2(\pi/2) + 4\sin \pi/2\right\} - \left\{0 - 0 + 0\right\}\right]$$

$$= \frac{a^2}{4}\left[-\pi - 0 + 4\right]$$

$$\left.\begin{matrix}\text{Required}\\ \text{Area}\end{matrix}\right\} = \frac{a^2}{4}\left[4 - \pi\right]$$

Triple Integrals and Volume of Solids :-

① Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$.

Soln :- Let $I = \int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$

$$= \int_0^1\int_0^1\int_0^1 e^x . e^y . e^z\,dx\,dy\,dz$$

$$= \int_0^1 e^x\,dx \times \int_0^1 e^y\,dy \times \int_0^1 e^z\,dz$$

$$= \left[e^x\right]_{x=0}^{x=1} \times \left[e^y\right]_{y=0}^{y=1} \times \left[e^z\right]_{z=0}^{z=1}$$

$$= (e^1 - e^0) \times (e^1 - e^0) \times (e^1 - e^0)$$

$$= (e^1 - e^0)^3$$

$$= (e^1 - 1)^3$$

$$= (e - 1)^3 \quad //$$

② Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz . dy . dx$

Soln:-

Let $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz . dy . dx$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz \right] dy . dx .$$

Now $\int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz = \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{A^2-z^2}} \, dz$

where $A = \sqrt{a^2-x^2-y^2}$

$$\boxed{\int \frac{1}{\sqrt{A^2-z^2}} dz = \sin^{-1}\left(\frac{z}{A}\right)}$$

$$\therefore \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz = \left[\sin^{-1}\left(\frac{z}{A}\right) \right]_{z=0}^{z=\sqrt{a^2-x^2-y^2}}$$

$$= \left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right) \right]_{z=0}^{z=\sqrt{a^2-x^2-y^2}}$$

$$= \sin^{-1}\left(\frac{\sqrt{a^2-x^2-y^2}}{\sqrt{a^2-x^2-y^2}}\right) - \sin^{-1}(0)$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \pi/2 - 0 .$$

$$\int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz = \pi/2 .$$

$$\therefore I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2} \, dy \, dx$$

$$= \frac{\pi}{2} \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} dy \right] dx .$$

$$= \frac{\pi}{2} \int_0^a \left(\sqrt{a^2-x^2}\right).\, dx$$

$$\boxed{\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}}$$

$$\therefore\ I = \frac{\pi}{2}\left[\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}\right]_{x=0}^{x=a.}$$

$$= \frac{\pi}{2}\left[\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right) + \frac{a}{2}\sqrt{a^2-a^2} - 0 - 0\right]$$

$$= \frac{\pi}{2}\left[\frac{a^2}{2} \times \sin^{-1}(1)\right]$$

$$= \frac{\pi}{2} \times \frac{a^2}{2} \times \frac{\pi}{2}$$

$$I. = \frac{\pi^2 a^2}{8}\ //.$$

③ Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$

Soln

Refer Sum ②

Write Sum ② , Replace a=1 .

④ $\int_0^{\log a} \int_0^{x} \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$ Evaluate

Soln

Let $I = \int_0^{\log a} \int_0^{x} \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$ [correct form]

$$= \int_0^{\log a} \int_0^{x} \left[\int_0^{x+y} e^{x} . e^{y} . e^{z}\, dz\right] dy.\, dx.$$

consider $\int_0^{x+y} e^x \cdot e^y \cdot e^z \, dz = e^x \cdot e^y \left[e^z \right]_{z=0}^{z=x+y}$

$$= e^x \cdot e^y \left(e^{x+y} - e^0 \right)$$

$$= e^x \cdot e^y \left(e^x \cdot e^y - 1 \right)$$

$$= e^{2x} \cdot e^{2y} - e^x \cdot e^y$$

$$= e^{2x+2y} - e^x \cdot e^y .$$

$$\therefore \quad I = \int_0^{\log a} \left[\int_0^x \left(e^{2x+2y} - e^x e^y \right) . dy \right] . dx .$$

Now $\int_0^x \left(e^{2x+2y} - e^x e^y \right) . dy = \left[\frac{e^{2x+2y}}{2} - e^x e^y \right]_{y=0}^{y=x}$

$$= \left[\left\{ \frac{e^{2x+2x}}{2} - e^x \cdot e^x \right\} - \left\{ \frac{e^{2x}}{2} - e^x \right\} \right]$$

$$= \frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x$$

$$= \frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x$$

$$\therefore \quad I = \int_0^{\log a} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^x \right) . dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x \right]_{x=0}^{x=\log a}$$

$$= \left[\frac{e^{4\log a}}{8} - \frac{3e^{2\log a}}{4} + e^{\log a} \right] - \left[\frac{e^0}{8} - \frac{3e^0}{4} + e^0 \right]$$

$$= \left[\frac{e^{\log a^4}}{8} - \frac{3e^{\log a^2}}{4} + e^{\log a} \right] - \left[\frac{1}{8} - \frac{3}{4} + 1 \right]$$

$$= \left[\frac{a^4}{8} - \frac{3a^2}{4} + a \right] - \left[\frac{1-6+8}{8} \right]$$

$$= \frac{a^4}{8} - \frac{3a^2}{4} + a - \left(\frac{3}{8}\right)$$

$$= \frac{a^4 - 6a^2 + 8a - 3}{8} \quad //$$

⑤ Find the volume of the portion of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which lies in the octant using triple integration.

Soln:-

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

x upp

$y = 0, z = 0$

$\frac{x^2}{a^2} = 1$,

$x^2 = a^2$

$x = a$.

y upp

$z = 0$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

$y = \frac{b}{a}\sqrt{1 - x^2/a^2}$

z upp.

	low	upper
z	0	$c\sqrt{1 - x^2/a^2 - y^2/b^2}$
y	0	$b\sqrt{1 - x^2/a^2}$
x	0	a

$$\therefore \text{Required Volume} = \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\, dy\, dx$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \Big[z \Big]_{z=0}^{z = c\sqrt{1-x^2/a^2-y^2/b^2}} dy\, dx$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - x^2/a^2 - y^2/b^2}\; dy\, dx.$$

$$= c \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\; dy\, dx.$$

$$= c \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right) - b^2\left(\frac{y^2}{b^2}\right)}{b^2}}\; dy\, dx \qquad \left(\begin{array}{c}\text{Here} \\ \times \text{ly and } \div \text{ by} \\ b^2\end{array}\right)$$

$$= \frac{c}{b} \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{b^2(1 - x^2/a^2) - y^2}\; dy\, dx$$

$$= \frac{c}{b}\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{A^2-y^2}\; dy\, dx \quad \text{where } A=\sqrt{b^2(1-x^2/a^2)}$$

Formula $$\boxed{\int \sqrt{A^2-y^2}\,dy = \frac{A^2}{2}\sin^{-1}\left(\frac{y}{A}\right)+\frac{y}{2}\sqrt{A^2-y^2}}$$

$$\therefore \text{ Required volume } = \frac{c}{b}\int_0^a \left[\frac{A^2}{2}\sin^{-1}\left(\frac{y}{A}\right)+\frac{y}{2}\sqrt{A^2-y^2}\right]_{y=0}^{y=b\sqrt{1-x^2/a^2}} dx$$

$$= \frac{c}{b}\int_0^a \left[\frac{b^2(1-x^2/a^2)}{2}\sin^{-1}\left(\frac{b\sqrt{1-x^2/a^2}}{b\sqrt{1-x^2/a^2}}\right)+0-0-0\right] dx$$

$$= \frac{c}{b}\times\frac{b^2}{2}\int_0^a \frac{a^2-x^2}{a^2}\times\frac{\pi}{2}.\; dx.$$

$$= \frac{cb}{2}\times\frac{\pi}{2}\int_0^a\left(\frac{a^2}{a^2}-\frac{x^2}{a^2}\right).dx$$

$$= \frac{\pi bc}{4}\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$$

$$= \frac{\pi bc}{4}\left[x-\frac{x^3}{3a^2}\right]_{x=0}^{x=a}$$

$$= \frac{\pi bc}{4}\left[a-\frac{a^3}{3a^2}\right]$$

$$= \frac{\pi bc}{4}\left[a-\frac{a}{3}\right]$$

$$\text{Req. volume} = \frac{\pi bc}{4}\left[\frac{2a}{3}\right] = \underline{\underline{\frac{\pi abc}{6}}}.$$

$$\text{Volume of whole ellipsoid} = 8\times\text{Volume in octant}$$
$$= 8\times\frac{\pi abc}{6}$$
$$= \frac{4\pi abc}{3}\;//.$$

⑥ Evaluate $\iiint\limits_V \frac{dz\,dy\,dx}{(x+y+z+1)^3}$ over the region of integration bounded by planes $x=0$, $y=0$, $z=0$, $x+y+z=1$

Soln.

	low	upp.
x	0	1
y	0	$1-x$
z	0	$1-x-y$

$$\therefore \quad I = \iiint\limits_V \frac{dz\,dy\,dx}{(x+y+z+1)^3} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz.dy.dx}{(x+y+z+1)^3}.$$

$$= \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3} \right] dy.dx.$$

Consider $\int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3} = \int_0^{1-x-y} (x+y+z+1)^{-3} dz$

$$= \left[\frac{(x+y+z+1)^{-3+1}}{(-3+1)} \right]_{z=0}^{z=1-x-y}$$

$$= \left[\frac{(x+y+z+1)^{-2}}{-2} \right]_{z=0}^{z=1-x-y}$$

$$= \left[\frac{(x+y+1-x-y+1)^{-2}}{-2} - \frac{(x+y+1)^{-2}}{-2} \right]$$

$$= \left[\frac{(2)^{-2}}{-2} + \frac{(x+y+1)^{-2}}{2} \right]$$

$$= \frac{1}{-2} \left[\frac{1}{4} - (x+y+1)^{-2} \right].$$

$$\therefore \; I = -\frac{1}{2}\int_0^1\int_0^{1-x}\left(\frac{1}{4} - (x+y+1)^{-2}\right) dy\, dx.$$

$$= -\frac{1}{2}\int_0^1\left[\frac{y}{4} - \frac{(x+y+1)^{-2+1}}{-1}\right]_{y=0}^{y=1-x} dx$$

$$= -\frac{1}{2}\int_0^1\left[\left\{\frac{(1-x)}{4} + (x+1+1-x)^{-1}\right\} - \left\{0 + \underline{(x+1)^{-1}}\right\}\right] dx$$

$$= -\frac{1}{2}\int_0^1\left(\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right). dx.$$

$$= -\frac{1}{2}\int_0^1\left(\frac{1}{4} - \frac{x}{4} + \frac{1}{2} - \frac{1}{x+1}\right). dx$$

$$= -\frac{1}{2}\int_0^1\left(\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1}\right). dx.$$

$$= -\frac{1}{2}\left[\frac{3}{4}x - \frac{x^2}{8} - \log(x+1)\right]_{x=0}^{x=1}$$

$$= -\frac{1}{2}\left[\left(\frac{3}{4} - \frac{1}{8} - \log(2)\right) - \left(0 - 0 - \log(1)\right)\right]$$

$$= -\frac{1}{2}\left(\frac{3}{4} - \frac{1}{8} - \log 2\right)$$

$$= -\tfrac{1}{2}\left(\frac{6-1}{8} - \log 2\right)$$

$$= -\tfrac{1}{2}\left(\frac{5}{8} - \log 2\right)$$

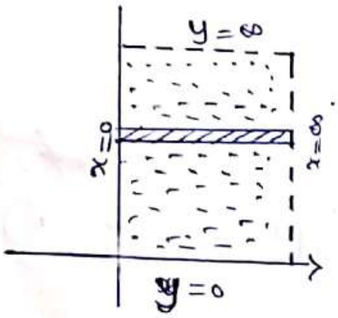
$$\boxed{I = \frac{1}{2}\log 2 - \frac{5}{16}}$$ Ans.

Changing into Polar coordinates :-

① By changing into polar co-ordinates show that $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \frac{\pi}{4}$. Hence evaluate $\int_0^\infty e^{-t^2} dt$.

Soln.

Let $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$.



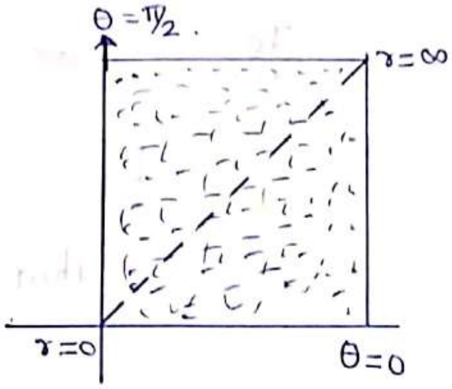
Changing into Polar coordinates

$\Longrightarrow$

$x = r\cos\theta$

$y = r\sin\theta$

$dxdy = r\,dr\,d\theta$



$$\therefore I = \int_0^{\pi/2} \int_0^\infty e^{-(r^2\cos^2\theta + r^2\sin^2\theta)} . r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\int_0^\infty e^{-r^2} r\, dr . \right] d\theta .$$

Put $r^2 = t$.

$2r . dr = dt$

$r\,dr = \frac{dt}{2}$.

r	0	∞
t	0	∞

$$= \int_0^{\pi/2} \left[\int_0^\infty e^{-t} \times \frac{dt}{2} \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\int_0^\infty e^{-t} dt \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\frac{e^{-t}}{-1} \right]_{t=0}^{t=\infty} d\theta .$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\frac{e^{-\infty}}{-1} - \frac{e^{0}}{-1} \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 1 \, d\theta$$

$$= \frac{1}{2} \left[\theta \right]_{\theta=0}^{\theta=\pi/2}$$

$$= \frac{1}{2} \times \frac{\pi}{2}$$

$$\boxed{I = \pi/4.}$$

To find : $\int_0^\infty e^{-t^2} . dt$

Let $A = \int_0^\infty e^{-t^2} dt$

Then $A = \int_0^\infty e^{-x^2} dx$ (Replace t by x)

Also $A = \int_0^\infty e^{-y^2} dy$ (Replace t by y)

$$A \cdot A = \int_0^\infty e^{-x^2} dx \times \int_0^\infty e^{-y^2} dy$$

Then $A \cdot A = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx . dy$

$A \cdot A = \pi/4$ (using above result)

$A^2 = \pi/4$.

$\therefore A = \frac{\sqrt{\pi}}{2}.$

$$\therefore \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

② Evaluate $\int_0^a \int_y^a \frac{x^2 \, dx \, dy}{(x^2+y^2)^{3/2}}$ by changing into polar coordinates.

Soln

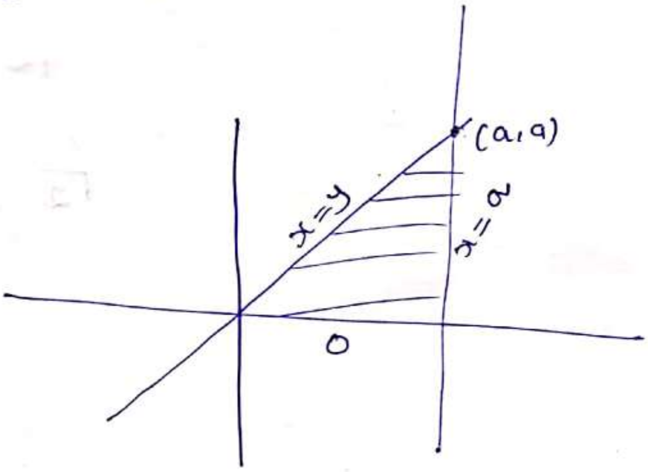
Take $I = \int_0^a \int_y^a \frac{x^2 \, dx \, dy}{(x^2+y^2)^{3/2}}$

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2$



After changing into polar co-ordinates

$$I = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{(r^2)^{3/2}} r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \cos^2\theta \; dr \, d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta \left[r \right]_0^{a\sec\theta} d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta \; a\sec\theta \; d\theta$$

$$= a\int_{0}^{\pi/4} \cos\theta \, d\theta$$

$$= a \Big[\sin\theta\Big]_{\theta=0}^{\theta=\pi/4}$$

$$= a \sin\pi/4$$

$$= a \times \frac{1}{\sqrt{2}}$$

$$= a/\sqrt{2} \;//$$

③ By changing into polar coordinates evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$

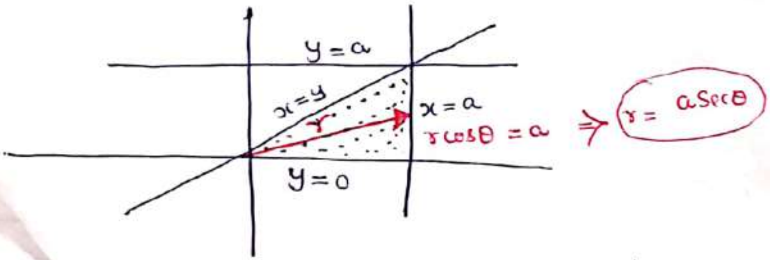
Soln

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dx\,dy = r\,dr\,d\theta \end{array}}$$

Take $I = \int_0^a \int_y^a \frac{x}{(x^2+y^2)}\,dx\,dy$

x varies from y to a

y varies from 0 to a



After changing into polar co-ordinates

r varies from 0 to $a\sec\theta$

θ varies from 0 to $\pi/4$.

$$\therefore I = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\, r.\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos\theta}{r^2}\,dr\,d\theta$$

$$= \int_0^{\pi/4} \Big[\cos\theta\ r\Big]_{r=0}^{r=a\sec\theta} d\theta$$

$$= \int_0^{\pi/4} (\cos\theta.\ a.\sec\theta - 0)\,d\theta$$

$$= a\int_0^{\pi/4} d\theta$$

$$= a\Big[\theta\Big]_{\theta=0}^{\theta=\pi/4}$$

$$= a\left(\pi/4 - 0\right)$$

$$= a\pi/4$$